Linear Algebra II
Exercise Sheet no. 2

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Exercise 1 (Further properties of eigenvalues/eigenspaces)
Throughout this exercise, A is a square matrix with entries in $\mathbb{R}$.

(a) Prove or disprove that A and A^t have the same eigenvalues. (A^t is the transpose of A, see lecture notes for Linear Algebra I.)

(b) Prove or disprove that A and A^t have the same eigenspaces.

(c) Assume A is regular and let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ. Show that $\mathbf{v}$ is also an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

(d) Let $\mathbf{v}$ be an eigenvector of the matrix A with eigenvalue λ and let s be a scalar. Show that $\mathbf{v}$ is an eigenvector of $A - sE$ with eigenvalue $\lambda - s$.

Exercise 2 (Application of diagonalisation: Fibonacci Numbers, Golden Mean)
Recall that the sequence $f_0, f_1, f_2, \dots$ of Fibonacci numbers is inductively defined as follows (cf Exercise H6.4 from LA I):

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_{k+2} &= f_{k+1} + f_k. \end{aligned}$$

(a) We define $\mathbf{u}_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} \in \mathbb{R}^2$. Find a matrix A such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$ for all $k \in \mathbb{N}$.

(b) What are the eigenvalues of A? Give an explicit formula for f_k.
Hint: Use the eigenvalues λ_1 and λ_2 as abbreviations as long as possible.

(c) Compute the limit $a = \lim_{k\to\infty} \frac{f_{k+1}}{f_k}$.
The limit is called the Golden Mean, it divides a line segment of length 1 into two parts a and $1-a$ such that $\frac{1}{a} = \frac{a}{1-a}$.

Exercise 3 (Euclidean algorithm and recursive functions)
The greatest common divisor $d = \gcd(a, b)$ of two non-zero natural numbers a and b is a natural number characterised by the following property:

- $d|b$ and $d|a$.
- If $r|a$ and $r|b$, then $r|d$

The Euclidean algorithm is a procedure for determining the greatest common divisor of two numbers a and b.

Step 0: Swap a and b if $a < b$.
Euclid (a,b): IF $b = 0$ THEN return a, ELSE return Euclid$(b, a \bmod b)$.

[After initialising: $d_1 := \min\{a, b\}, d_0 := \max\{a, b\}$, we divide with remainder in each step: $d_{k-1} = q_k d_k + d_{k+1}$ with $0 \le d_{k+1} < d_k$, and ends if $d_{k+1} = 0$. We get $d_k = \gcd(a, b)$.]

(a) Prove that the gcd is well defined, that is, uniquely characterised by the above properties.

(b) Prove that $\gcd(a, b) = \gcd(b, a \bmod b)$ for $0 < b$.

(c) Assume that $b < a$ and that $(a,b) \xrightarrow{E} (a',b') \xrightarrow{E} (a'',b'')$ are two steps in the Euclidean algorithm. Prove that $a'' < \frac{a}{2}$ and $b'' < \frac{b}{2}$.

(d) Deduce that Euclid(a,b) returns gcd(a,b).

Exercise 4 (Diagonalization and recursive sequences)

Let a_k be the sequence of real numbers defined recursively as follows: $a_0 = 0$, $a_1 = 1$, and $a_{k+2} = \frac{1}{2}(a_{k+1} + a_k)$. In other words, each term in the sequence is the average of the two previous terms.

(a) As we did with the Fibonacci sequence, we want to study this sequence using diagonalization of matrices. For $k \geq 0$, let

$$\mathbf{u}_k = \begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix}.$$

Using the equations $a_{k+2} = \frac{1}{2}(a_{k+1} + a_k)$ and $a_{k+1} = a_{k+1}$, find a 2×2 matrix A such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

(b) Find the eigenvalues and eigenvectors of A, and find a matrix S and a diagonal matrix D with $D = S^{-1}AS$.

(c) Find a formula for a_k, and calculate $\lim_{k\to\infty} a_k$ if it exists.