Linear Algebra II
Exercise Sheet no. 1

TECHNISCHE UNIVERSITÄT DARMSTADT

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

SS 2011
April 11, 2011

Exercise G1 (Warm-up)

In $\mathbb{R}^3$, let g be a line through the origin and E be a plane through the origin such that g is not in E. Determine (geometrically) the eigenvalues and eigenspaces of the following linear maps:

(a) reflection in the plane E.

(b) central reflection in the origin.

(c) parallel projection in the direction of g onto E.

(d) rotation about g through $\frac{1}{3}\pi$ followed by rescaling in the direction of g with factor 6.

Which of these maps admit a basis of eigenvectors?

Exercise G2 (Warm-up)

(a) Suppose that $\varphi : V \to V$ is a linear map over an arbitrary field, and such that all vectors $\mathbf{v} \in \mathbf{V}$ are eigenvectors of φ. Show that φ must have exactly one eigenvalue λ, and that φ is precisely $\lambda \cdot \mathrm{id}$, where id is the identity map.

(b) Let $\psi : \mathbb{R}^4 \to \mathbb{R}^4$ be the map defined by

$$\varphi \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ -w \\ z \end{pmatrix}.$$

Find the (real) eigenvalues of φ and their multiplicity, and find bases for the corresponding eigenspaces.

Exercise G3 (Fixed points of affine maps)

Recall that an affine map is a function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\varphi(\mathbf{x}) = \varphi_0(\mathbf{x}) + \mathbf{b}$ where φ_0 is a linear map and $\mathbf{b} \in \mathbb{R}^2$ is a vector. In this exercise we are interested in the question of whether such a map φ has a *fixed point,* i.e., a point $\mathbf{x}$ such that $\varphi(\mathbf{x}) = \mathbf{x}$.

(a) Prove that φ has a fixed point, provided that 1 is not an eigenvalue of φ_0.

(b) Let φ be a rotation through the angle α about a point $\mathbf{c}$. Give a formula for φ w.r.t. the standard basis, i.e., find functions f and g such that $\varphi(x, y) = (f(x, y), g(x, y))$.

(c) Let $\varrho_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation through the angle α (about the origin) and let $\tau_{\mathbf{c}} : \mathbf{x} \mapsto \mathbf{x} + \mathbf{c}$ be the translation by $\mathbf{c}$. Using (ii), show that the composition $\tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}}$ is a rotation through α about the point $\mathbf{c}$.

(d) Suppose that the linear map φ_0 is a rotation through an angle $\alpha \neq 0$. Prove that the affine map $\varphi : \mathbf{x} \mapsto \varphi_0(\mathbf{x}) + \mathbf{b}$ has a fixed point $\mathbf{c}$ and that $\varphi = \tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}}$, i.e., φ is a rotation through α about $\mathbf{c}$.
(Bonus question: how can you find the centre $\mathbf{c}$ *geometrically* (i.e., without computation)?)

(e) Give an example of an affine map $\varphi(\mathbf{x}) = \varphi_0(\mathbf{x}) + \mathbf{b}$ without fixed points such that φ_0 is not the identity map.

Exercise G4 (Eigenvalues and eigenvectors)

Consider the real 2×2 matrix $A = \begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix}$ and the linear map $\varphi = \varphi_A$ given by A w.r.t. the standard basis.

(a) Calculate the eigenvalues of A by expanding $\det(A - \lambda E)$ and find the zeroes/roots of the characteristic polynomial.

(b) For each eigenvalue λ_i determine the eigenspace V_{λ_i}.

(c) Find a basis B of $\mathbb{R}^2$ that only consists of eigenvectors of φ and find the matrix of the map φ with respect to the basis B.