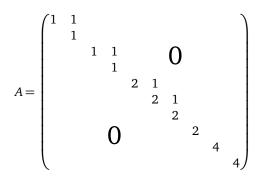
# Linear Algebra II Exercise Sheet no. 15



TECHNISCHE UNIVERSITÄT DARMSTADT

Prof. Dr. Otto Dr. Le Roux Dr. Linshaw

**Exercise 1** (Characteristic and minimal polynomials) Find the characteristic and minimal polynomials of the following matrix



Solution:

The characteristic polynomial is

$$p_A(x) = (1-x)^4 (2-x)^4 (4-x)^2$$
,

and the minimal polynomial is

$$q_A(x) = (x-1)^2(x-2)^3(x-4).$$

## Exercise 2 (Jordan normal form)

(a) Let  $\varphi$  be an endomorphism of a ten-dimensional  $\mathbb{F}$ -vector space *V*. W.r.t. basis  $B = (\mathbf{b}_1, \dots, \mathbf{b}_{10})$  let  $\varphi$  be represented by a Jordan normal form matrix with three Jordan blocks for the same eigenvalue  $\lambda \in \mathbb{F}$ , of sizes 2, 3 and 5. Let  $\psi := \varphi - \lambda id$ . Complete the following table:

i	1	2	3	4	5	6	7	8	9	10
dim(ker $\psi^i$ )					10	10	10	10	10	10

In the notation of Lemma 1.6.4 of the notes: for which  $\mathbf{v} \in V$  does  $[\![\mathbf{v}]\!]$  have maximal dimension? Split the basis *B* in a way to obtain bases for the two invariant subspaces  $V = [\![\mathbf{v}]\!] \oplus V'$  (as in Claim 1.6.5). If  $\varphi'$  is the restriction of  $\varphi$  to *V*', what is the matrix representation of  $\varphi'$  with respect to this basis? If  $\psi' = \varphi' - \lambda id$ , how is the corresponding table for  $\psi'$  related to the above?

(b) Now, let  $\varphi$  be another endomorphism of *V* with characteristic polynomial  $(\lambda - X)^{10}$ . Suppose we have the following data for  $\psi = \varphi - \lambda id$ :

i	1	2	3	4	5	6	7	8	9	10
dim(ker $\psi^i$ )	3	5	7	8	9	10	10	10	10	10

Determine the Jordan normal form representation of  $\varphi$  from this data (up to permutation of Jordan blocks).

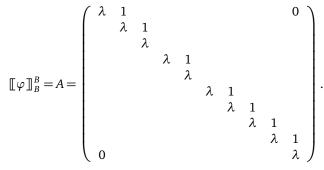
(c) (extra) In general, let  $\varphi_0$  and  $\varphi_1$  be two endomorphisms of  $\mathbb{F}$ -vector spaces  $V_0$  and  $V_1$  of the same finite dimension, with the same characteristic polynomial that splits into linear factors. Suppose moreover that for each eigenvalue  $\lambda$  of  $\varphi_0$  and  $\varphi_1$ , the tables for  $\psi_0 = \varphi_0 - \lambda$  id and  $\psi_1 = \varphi_1 - \lambda$  id are the same.

Sketch a proof for the similarity of  $\varphi_0$  and  $\varphi_1$  adapting the argument for the existence and uniqueness of the Jordan normal form. How can this be used to give a "different" proof for the similarity of *A* and *A*<sup>t</sup> for any matrix  $A \in \mathbb{C}^{(n,n)}$ ?

Summer term 2011 July 13, 2011

### Solution:

a) Let



The entries of the table are:

i	1	2	3	4	5	6	7	8	9	10
dim(ker $\psi^i$ )	3	6	8	9	10	10	10	10	10	10

For  $\mathbf{v} = \lambda \mathbf{b}_{10}$  with  $\lambda \neq 0$ , does  $[\![\mathbf{v}]\!]$  have maximal dimension 5. The natural choice for a basis for V' is  $(\mathbf{b}_1, \dots, \mathbf{b}_5)$ , and the representation of  $\varphi'$  with respect to this basis is the 5 × 5-matrix in the top left corner of A:

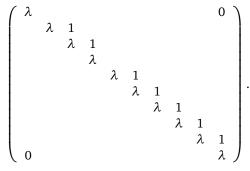
$$A = \left( \begin{array}{cccc} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{array} \right).$$

The table for  $\psi'$  has the form:

i	1	2	3	4	5
$\dim(\ker \psi'^i)$	2	4	5	5	5

The *i*th entry of this table is *i* less than the entry of the previous table, until we reach the dimension of [v].

b) The Jordan normal form is:



When  $\mathbf{b}_i \in \ker \psi^{j+1} \setminus \ker \psi^j$ , then *j* depends as follows on *i*:

	i	1	2	3	4	5	6	7	8	9	10
[.	j	1	1	2	3	1	2	3	4	5	6

c) Sketch: we can without loss of generality restrict ourselves to the case that  $\varphi_0$  and  $\varphi_1$  have  $\lambda$  as sole eigenvalue. Then we proceed by induction on dim  $V_0 = \dim V_1$ . By assumption, the tables for  $\lambda$  are the same for both  $\psi_0 = \varphi_0 - \lambda id$  and  $\psi_1 = \varphi_1 - \lambda id$ . This means the first j such that dim $(\ker \psi_0^j) = \dim V_0$  is also the first j such that dim $(\ker \psi_1^j) = \dim V_1$ . Then there are vectors  $\mathbf{v}_0 \in \ker \psi_0^{j+1} \setminus \ker \psi_0^j$  and  $\mathbf{v}_1 \in \ker \psi_1^{j+1} \setminus \ker \psi_1^j$ , allowing us to split up  $V_0$  and  $V_1$  as  $V_0 = [\![\mathbf{v}_0]\!] \oplus V'_0$  and  $V' = [\![\mathbf{v}_1]\!] \oplus V'_1$ . We let  $\varphi'_0$  and  $\varphi'_1$  be the restrictions of  $\varphi_0$  and  $\varphi_1$  to  $V'_0$  and  $V'_1$  respectively. We know that  $\varphi'_0$  and  $\varphi'_1$  also have  $\lambda$  as sole eigenvalue!, their tables are obtained in the same manner from those of  $\varphi_0$  and  $\varphi_1$ , and therefore  $\varphi'_0$  and  $\varphi'_1$  are represented by similar matrices by induction hypothesis. This can be extended to an isomorphism of  $V_0$  and  $V_1$  showing that  $\varphi_0$  and  $\varphi_1$  are represented by similar matrices.

Since *A* and *A*<sup>t</sup> have the same characteristic polynomial, and dim $(\ker(A - \lambda E)^i) = \dim(\ker(A^t - \lambda E)^i)$  for every eigenvalue  $\lambda$  and  $i \in \mathbb{N}$  (for a matrix has the same rank as its transpose), we deduce that every matrix is similar to its transpose over  $\mathbb{C}$ .

Exercise 3 (Diagonalization using orthogonal matrices)

Let  $\varphi$  be the endomorphism of  $\mathbb{R}^3$  given in the standard basis by

$$A = \begin{pmatrix} 1 & -1 & -1/\sqrt{2} \\ -1 & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 2 \end{pmatrix}.$$

- (a) Find an orthogonal matrix *B* such that  $B^{-1}AB$  is diagonal.
- (b) Describe all subspaces of  $\mathbb{R}^3$  which are invariant under  $\varphi$ .
- (c) For  $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ , let *Q* be the quadratic form

$$Q(\mathbf{x}) = x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_2 - \frac{2}{\sqrt{2}}x_1x_3 + \frac{2}{\sqrt{2}}x_2x_3.$$

Find the principle axes of the quadric *X* given by  $Q(\mathbf{x}) = 1$ .

#### Solution:

a) The characteristic polynomial of *A* is -(x - 1)(x - 3)(x), and the normalized eigenvectors of eigenvalues 3, 1, and 0, respectively, are

$$\mathbf{v}_{1} = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{v}_{2} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{v}_{3} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$
  
Therefore the orthogonal matrix  $B = \begin{pmatrix} -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$  satisfies  $B^{-1}AB = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$ 

- b) The invariant subspaces are  $\{0\}$ , span $(v_1)$ , span $(v_2)$ , span $(v_3)$ , span $(v_1, v_2)$ , span $(v_1, v_3)$ , span $(v_2, v_3)$ , and  $\mathbb{R}^3$ .
- c) Since the matrix representing Q is precisely A, the principal axes of X lie along the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

#### **Exercise 4** (Invariant planes in $\mathbb{R}^4$ )

Let  $\varphi$  be an orthogonal transformation of  $\mathbb{R}^4$  which fixes a plane  $U_1$  pointwise, and acts by a nontrivial rotation on another plane  $U_2$ . Prove that  $U_1$  and  $U_2$  are the only invariant subspaces of  $\mathbb{R}^4$  of dimension 2.

#### Solution:

First, we have  $U_1 \cap U_2 = \{\mathbf{0}\}$ , since this is the only vector that is fixed pointwise under a nontrivial rotation. Therefore we have a direct sum decomposition  $\mathbb{R}^4 = U_1 \oplus U_2$ . Suppose that *V* is a two-dimensional invariant subspace of  $\mathbb{R}^4$ , and  $V \neq U_1$  and  $V \neq U_2$ . Fix a non-null vector  $\mathbf{v} \in V$ ; we have a unique decomposition  $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$ , with  $\mathbf{u}_i \in U_i$ . Since  $V \neq U_1$ , we may choose  $\mathbf{v}$  so that  $\mathbf{u}_2 \neq \mathbf{0}$ .

First, we claim that  $\mathbf{u}_1 \neq \mathbf{0}$ . Otherwise, we would have  $\mathbf{v} \in U_2$ , and hence  $V = U_2$  since  $\mathbf{v}$  and  $\varphi(\mathbf{v})$  are linearly independent. (This follows from the fact that  $\varphi$  acts by a non-trivial rotation on  $U_2$ ). Then

$$\varphi(\mathbf{v}) = \varphi(\mathbf{u}_1) + \varphi(\mathbf{u}_2) = \mathbf{u}_1 + \varphi(\mathbf{u}_2)$$

Since  $\mathbf{u}_2 \neq \mathbf{0}$ , we have  $\mathbf{v} - \varphi(\mathbf{v}) = \mathbf{u}_2 - \varphi(\mathbf{u}_2) \neq \mathbf{0}$ . Since *V* is invariant under  $\varphi$ , it follows that  $\mathbf{u}_2 - \varphi(\mathbf{u}_2)$  lies in *V*. But  $\mathbf{u}_2 - \varphi(\mathbf{u}_2)$  also lies in  $U_2$ . By applying  $\varphi$  again, we see that  $\mathbf{u}_2 - \varphi(\mathbf{u}_2)$  and  $\varphi(\mathbf{u}_2 - \varphi(\mathbf{u}_2))$  span  $U_2$ , so we have  $V = U_2$ , which is a contradiction.