Linear Algebra II
Exercise Sheet no. 15

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
July 13, 2011

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Exercise 1 (Characteristic and minimal polynomials)
Find the characteristic and minimal polynomials of the following matrix

$$A = \begin{pmatrix} 1 & 1 & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & 1 & & & \text{\Large 0} & & & \\ & & & 1 & & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & 1 & & & \\ & & & & & & 2 & & & \\ & & & \text{\Large 0} & & & & 2 & & \\ & & & & & & & & 4 & \\ & & & & & & & & & 4 \end{pmatrix}$$

Solution:
The characteristic polynomial is

$$p_A(x) = (1-x)^4(2-x)^4(4-x)^2,$$

and the minimal polynomial is

$$q_A(x) = (x-1)^2(x-2)^3(x-4).$$

Exercise 2 (Jordan normal form)

(a) Let φ be an endomorphism of a ten-dimensional $\mathbb{F}$-vector space V. W.r.t. basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_{10})$ let φ be represented by a Jordan normal form matrix with three Jordan blocks for the same eigenvalue $\lambda \in \mathbb{F}$, of sizes 2, 3 and 5. Let $\psi := \varphi - \lambda\mathrm{id}$. Complete the following table:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$					10	10	10	10	10	10

In the notation of Lemma 1.6.4 of the notes: for which $\mathbf{v} \in V$ does $[\![\mathbf{v}]\!]$ have maximal dimension? Split the basis B in a way to obtain bases for the two invariant subspaces $V = [\![\mathbf{v}]\!] \oplus V'$ (as in Claim 1.6.5). If φ' is the restriction of φ to V', what is the matrix representation of φ' with respect to this basis? If $\psi' = \varphi' - \lambda\mathrm{id}$, how is the corresponding table for ψ' related to the above?

(b) Now, let φ be another endomorphism of V with characteristic polynomial $(\lambda - X)^{10}$. Suppose we have the following data for $\psi = \varphi - \lambda\mathrm{id}$:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$	3	5	7	8	9	10	10	10	10	10

Determine the Jordan normal form representation of φ from this data (up to permutation of Jordan blocks).

(c) (extra) In general, let φ_0 and φ_1 be two endomorphisms of $\mathbb{F}$-vector spaces V_0 and V_1 of the same finite dimension, with the same characteristic polynomial that splits into linear factors. Suppose moreover that for each eigenvalue λ of φ_0 and φ_1, the tables for $\psi_0 = \varphi_0 - \lambda\mathrm{id}$ and $\psi_1 = \varphi_1 - \lambda\mathrm{id}$ are the same.
Sketch a proof for the similarity of φ_0 and φ_1 adapting the argument for the existence and uniqueness of the Jordan normal form. How can this be used to give a "different" proof for the similarity of A and A^t for any matrix $A \in \mathbb{C}^{(n,n)}$?

Solution:

a) Let

$$[\![\varphi]\!]_B^B = A = \begin{pmatrix} \lambda & 1 & & & & & & & & 0 \\ & \lambda & 1 & & & & & & & \\ & & \lambda & & & & & & & \\ & & & \lambda & 1 & & & & & \\ & & & & \lambda & & & & & \\ & & & & & \lambda & 1 & & & \\ & & & & & & \lambda & 1 & & \\ & & & & & & & \lambda & 1 & \\ & & & & & & & & \lambda & 1 \\ 0 & & & & & & & & & \lambda \end{pmatrix}.$$

The entries of the table are:

i	1	2	3	4	5	6	7	8	9	10
$\dim(\ker \psi^i)$	3	6	8	9	10	10	10	10	10	10

For $\mathbf{v} = \lambda \mathbf{b}_{10}$ with $\lambda \neq 0$, does $[\![\mathbf{v}]\!]$ have maximal dimension 5. The natural choice for a basis for V' is $(\mathbf{b}_1, \ldots, \mathbf{b}_5)$, and the representation of φ' with respect to this basis is the 5×5-matrix in the top left corner of A:

$$A = \begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

The table for ψ' has the form:

i	1	2	3	4	5
$\dim(\ker \psi'^i)$	2	4	5	5	5

The ith entry of this table is i less than the entry of the previous table, until we reach the dimension of $[\![\mathbf{v}]\!]$.

b) The Jordan normal form is:

$$\begin{pmatrix} \lambda & & & & & & & & & 0 \\ & \lambda & 1 & & & & & & & \\ & & \lambda & 1 & & & & & & \\ & & & \lambda & & & & & & \\ & & & & \lambda & 1 & & & & \\ & & & & & \lambda & 1 & & & \\ & & & & & & \lambda & 1 & & \\ & & & & & & & \lambda & 1 & \\ & & & & & & & & \lambda & 1 \\ 0 & & & & & & & & & \lambda \end{pmatrix}.$$

When $\mathbf{b}_i \in \ker \psi^{j+1} \backslash \ker \psi^j$, then j depends as follows on i:

i	1	2	3	4	5	6	7	8	9	10
j	1	1	2	3	1	2	3	4	5	6

c) Sketch: we can without loss of generality restrict ourselves to the case that φ_0 and φ_1 have λ as sole eigenvalue. Then we proceed by induction on $\dim V_0 = \dim V_1$. By assumption, the tables for λ are the same for both $\psi_0 = \varphi_0 - \lambda \mathrm{id}$ and $\psi_1 = \varphi_1 - \lambda \mathrm{id}$. This means the first j such that $\dim(\ker \psi_0^j) = \dim V_0$ is also the first j such that $\dim(\ker \psi_1^j) = \dim V_1$. Then there are vectors $\mathbf{v}_0 \in \ker \psi_0^{j+1} \backslash \ker \psi_0^j$ and $\mathbf{v}_1 \in \ker \psi_1^{j+1} \backslash \ker \psi_1^j$, allowing us to split up V_0 and V_1 as $V_0 = [\![\mathbf{v}_0]\!] \oplus V_0'$ and $V' = [\![\mathbf{v}_1]\!] \oplus V_1'$. We let φ_0' and φ_1' be the restricitions of φ_0 and φ_1 to V_0' and V_1' respectively. We know that φ_0' and φ_1' also have λ as sole eigenvalue! , their tables are obtained in the same manner from those of φ_0 and φ_1, and therefore φ_0' and φ_1' are represented by similar matrices by induction hypothesis. This can be extended to an isomorphism of V_0 and V_1 showing that φ_0 and φ_1 are represented by similar matrices, by sending $\mathbf{v}_0$ to $\mathbf{v}_1$, $\varphi_0 \mathbf{v}_0$ to $\varphi_1 \mathbf{v}_1$, etcetera. This completes the (sketch of the) proof.

Since A and A^t have the same characteristic polynomial, and $\dim(\ker(A - \lambda E)^i) = \dim(\ker(A^t - \lambda E)^i)$ for every eigenvalue λ and $i \in \mathbb{N}$ (for a matrix has the same rank as its transpose), we deduce that every matrix is similar to its transpose over $\mathbb{C}$.

Exercise 3 (Diagonalization using orthogonal matrices)

Let φ be the endomorphism of $\mathbb{R}^3$ given in the standard basis by

$$A = \begin{pmatrix} 1 & -1 & -1/\sqrt{2} \\ -1 & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 2 \end{pmatrix}.$$

(a) Find an orthogonal matrix B such that $B^{-1}AB$ is diagonal.

(b) Describe all subspaces of $\mathbb{R}^3$ which are invariant under φ.

(c) For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, let Q be the quadratic form

$$Q(\mathbf{x}) = x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_2 - \frac{2}{\sqrt{2}}x_1x_3 + \frac{2}{\sqrt{2}}x_2x_3.$$

Find the principle axes of the quadric X given by $Q(\mathbf{x}) = 1$.

Solution:

a) The characteristic polynomial of A is $-(x-1)(x-3)(x)$, and the normalized eigenvectors of eigenvalues 3, 1, and 0, respectively, are

$$\mathbf{v}_1 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Therefore the orthogonal matrix $B = \begin{pmatrix} -1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$ satisfies $B^{-1}AB = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

b) The invariant subspaces are $\{\mathbf{0}\}$, span($\mathbf{v}_1$), span($\mathbf{v}_2$), span($\mathbf{v}_3$), span($\mathbf{v}_1, \mathbf{v}_2$), span($\mathbf{v}_1, \mathbf{v}_3$), span($\mathbf{v}_2, \mathbf{v}_3$), and $\mathbb{R}^3$.

c) Since the matrix representing Q is precisely A, the principal axes of X lie along the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Exercise 4 (Invariant planes in $\mathbb{R}^4$)

Let φ be an orthogonal transformation of $\mathbb{R}^4$ which fixes a plane U_1 pointwise, and acts by a nontrivial rotation on another plane U_2. Prove that U_1 and U_2 are the only invariant subspaces of $\mathbb{R}^4$ of dimension 2.

Solution:

First, we have $U_1 \cap U_2 = \{\mathbf{0}\}$, since this is the only vector that is fixed pointwise under a nontrivial rotation. Therefore we have a direct sum decomposition $\mathbb{R}^4 = U_1 \oplus U_2$. Suppose that V is a two-dimensional invariant subspace of $\mathbb{R}^4$, and $V \neq U_1$ and $V \neq U_2$. Fix a non-null vector $\mathbf{v} \in V$; we have a unique decomposition $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$, with $\mathbf{u}_i \in U_i$. Since $V \neq U_1$, we may choose $\mathbf{v}$ so that $\mathbf{u}_2 \neq \mathbf{0}$.

First, we claim that $\mathbf{u}_1 \neq \mathbf{0}$. Otherwise, we would have $\mathbf{v} \in U_2$, and hence $V = U_2$ since $\mathbf{v}$ and $\varphi(\mathbf{v})$ are linearly independent. (This follows from the fact that φ acts by a non-trivial rotation on U_2). Then

$$\varphi(\mathbf{v}) = \varphi(\mathbf{u}_1) + \varphi(\mathbf{u}_2) = \mathbf{u}_1 + \varphi(\mathbf{u}_2).$$

Since $\mathbf{u}_2 \neq \mathbf{0}$, we have $\mathbf{v} - \varphi(\mathbf{v}) = \mathbf{u}_2 - \varphi(\mathbf{u}_2) \neq \mathbf{0}$. Since V is invariant under φ, it follows that $\mathbf{u}_2 - \varphi(\mathbf{u}_2)$ lies in V. But $\mathbf{u}_2 - \varphi(\mathbf{u}_2)$ also lies in U_2. By applying φ again, we see that $\mathbf{u}_2 - \varphi(\mathbf{u}_2)$ and $\varphi(\mathbf{u}_2 - \varphi(\mathbf{u}_2))$ span U_2, so we have $V = U_2$, which is a contradiction.