Linear Algebra II
Exercise Sheet no. 14

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
July 30, 2011

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Exercise 1 (Bijection between conic sections)

Let $\mathbb{X}$ be the standard cone in $\mathbb{R}^3$ (defined by the equation $x_1^2 + x_2^2 = x_3^2$), let $\mathbb{A}_1, \mathbb{A}_2$ be the planes defined by the equations $x_3 = 1$ and $x_1 = \frac{1}{3}x_3 + \frac{2}{3}$, respectively and let L be the line that goes through the vectors $\begin{pmatrix} \frac{1}{3} \\ \pm\frac{2}{3} \\ 1 \end{pmatrix}$.

(a) Let $\mathbf{n} := \begin{pmatrix} -1 \\ 0 \\ \frac{1}{3} \end{pmatrix}$. Then the map

$$\varphi : \mathbb{A}_1 \setminus L \to \mathbb{A}_2$$

$$\mathbf{v} \mapsto -\frac{2}{3} \frac{1}{\langle \mathbf{n}, \mathbf{v} \rangle} \mathbf{v}$$

describes the central projection through the origin from $\mathbb{A}_1 \setminus L$ into $\mathbb{A}_2$. Make a sketch to verify this. (You only need to draw the (x_1, x_3)-plane.) Determine the image of φ.

(b) Sketch the conic sections that you get from $\mathbb{A}_i$ and $\mathbb{X}$, $i = 1, 2$. (You only need to draw the (x_1, x_3)-plane.)

(c) Compute a parametric description of $\mathbb{A}_1 \cap \mathbb{X}$ and $\mathbb{A}_2 \cap \mathbb{X}$.

(d) How can you extend φ to a bijection from the one conic section $(\mathbb{A}_1 \cap \mathbb{X})$ onto a completion of the other?

Solution:

a) We can write the equation for $\mathbb{A}_2$ as

$$\mathbb{A}_2 = \{\mathbf{v} : \langle \mathbf{v}, \mathbf{n} \rangle = -\frac{2}{3}\}.$$

First we show that φ is injective. Let $\mathbf{v}, \mathbf{w} \in \mathbb{A}_1 \setminus L$ be vectors with $\varphi(\mathbf{v}) = \varphi(\mathbf{w})$. Then we get

$$-\frac{2}{3} \frac{1}{\langle \mathbf{n}, \mathbf{v} \rangle} \mathbf{v} = -\frac{2}{3} \frac{1}{\langle \mathbf{n}, \mathbf{w} \rangle} \mathbf{w},$$

which implies

$$\mathbf{v} = \frac{\langle \mathbf{n}, \mathbf{v} \rangle}{\langle \mathbf{n}, \mathbf{w} \rangle} \mathbf{w}.$$

Note that $\langle \mathbf{n}, \mathbf{v} \rangle$ is non-zero on $\mathbb{A}_1 \setminus L$. As $\mathbf{v}$ and $\mathbf{w}$ lie in $\mathbb{A}_1$ they have to be equal.

b)

c) One parametric description for $\mathbb{X}$ is $\mathbb{X} = \{(r\sin\alpha, r\cos\alpha, r) : r, \alpha \in \mathbb{R}\}$. For the plane $\mathbb{A}_1$ we have $\mathbb{A}_1 = \{(x, y, 1) : x, y \in \mathbb{R}\}$ so that $\mathbb{A}_1 \cap \mathbb{X}$ has the paramteric description $\mathbb{A}_1 \cap \mathbb{X} = \{(\sin\alpha, \cos\alpha, 1) : \alpha \in \mathbb{R}\}$. To compute $\mathbb{A}_2 \cap \mathbb{X}$ we use the map φ on the set $\{(\sin\alpha, \cos\alpha, 1) : \alpha \in \mathbb{R}, \sin\alpha \neq \frac{1}{3}\}$:

$$\varphi(\begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 1 \end{pmatrix}) = -\frac{2}{3} \frac{1}{\langle \begin{pmatrix} -1 \\ 0 \\ \frac{1}{3} \end{pmatrix} \begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 1 \end{pmatrix} \rangle} \begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 1 \end{pmatrix} = \frac{2}{3\sin\alpha - 1} \begin{pmatrix} \sin\alpha \\ \cos\alpha \\ 1 \end{pmatrix}$$

It follows that $\mathbb{A}_2 \cap X$ has the parametric description:

$$\left\{ \left(\frac{2\sin\alpha}{3\sin\alpha - 1}, \frac{2\cos\alpha}{3\sin\alpha - 1}, \frac{2}{3\sin\alpha - 1} \right) : \alpha \in \mathbb{R}, \sin\alpha \neq \frac{1}{3} \right\}.$$

d) φ maps each $\mathbf{v} \in (\mathbb{A}_1 \setminus L) \cap \mathbb{X}$ onto the other conic section, because the line through $\mathbf{v}$ and $\varphi(\mathbf{v})$ lies on $\mathbb{X}$. Now we have still two points left: $\left(\frac{1}{3}, \pm\frac{2}{3}\sqrt{2}, 1\right)$. These we map to those points of the line at infinity that are represented by the asmyptotics of the hyperbola.

Exercise 2 (Minkowski space)

Consider the "Minkowski metric" on $\mathbb{R}^4$ induced by the symmetric bilinear form σ with diagonal entries $(1, 1, 1, -1)$ w.r.t. the standard basis.
The quadric $Q = \{\mathbf{v} \in \mathbb{R}^4 : \sigma(\mathbf{v}, \mathbf{v}) = 0\}$ is called the *null set* of σ.

(a) Show that σ is non-degenerate but has a non-trivial null set; determine the null set and describe it geometrically.

(b) Give examples of other bases of $\mathbb{R}^4$ w.r.t. which σ is represented by the matrix with diagonal entries $(1, 1, 1, -1)$, but which are not orthonormal w.r.t. the standard scalar product.

(c) Give an example of a subspace $U \subseteq \mathbb{R}^4$ s.t. $\mathbb{R}^4 \neq U \oplus U^\perp$ where

$$U^\perp := \{\mathbf{v} \in \mathbb{R}^4 : \sigma(\mathbf{v}, \mathbf{u}) = 0 \text{ for all } \mathbf{u} \in U\}$$

(d) Which are the signatures of the quadratic forms induced by σ on the 3-dimensional subspaces $U \subseteq \mathbb{R}^4$? Try to describe in each case the relation between the subspace U and the null set.

Solution:

a) For $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \neq \mathbf{0}$ we get $\sigma(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ -v_4 \end{pmatrix}) = v_1^2 + v_2^2 + v_3^2 + v_4^2 \neq 0.$

On the other hand, we have $\sigma(\mathbf{v}, \mathbf{v}) = 0$ for each vector $\mathbf{v}$ in the subspace spanned by $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. So the null set is not empty. It is a cone given by the equation $x_1^2 + x_2^2 + x_3^2 = x_4^2$. One gets this cone by taking one of the lines $\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ and $\lambda \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and rotating this set about the x_4-axis.

b) We choose the basis $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{5}{3} \\ -\frac{4}{3} \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$. Then

$$\sigma(\mathbf{v}_i, \mathbf{v}_j) = \sigma(\mathbf{v}_j, \mathbf{v}_i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{else}, \end{cases}$$

for $i \in \{1,2\}$ and $j \in \{1,2,3,4\}$. Furthermore we have

$$\sigma(\mathbf{v}_3,\mathbf{v}_4) = \frac{5}{3}\left(-\frac{4}{3}\right) + \frac{4}{3}\cdot\frac{5}{3} = 0 = \sigma(\mathbf{v}_4,\mathbf{v}_3).$$

and

$$\sigma(\mathbf{v}_3,\mathbf{v}_3) = \frac{25}{9} - \frac{16}{9} = 1, \quad \sigma(\mathbf{v}_4,\mathbf{v}_4) = \frac{16}{9} - \frac{25}{9} = -1.$$

Thus σ is represented by the matrix with diagonal entries $(1,1,1,-1)$ w.r.t this basis and the standard scalar product of $\mathbf{v}_3$ and $\mathbf{v}_4$ is

$$\langle \mathbf{v}_3,\mathbf{v}_4\rangle = \frac{5}{3}\left(-\frac{4}{3}\right) + \left(-\frac{4}{3}\right)\frac{5}{3} \neq 0.$$

c) Let U be the subspace spanned by $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3+\mathbf{e}_4$. $U^\perp = \{\mathbf{v}\in\mathbb{R}^4 : \sigma(\mathbf{v},\mathbf{w}) = 0 \text{ for all } \mathbf{u}\in U\}$. Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_3 \end{pmatrix}$ be an element in $U^\perp$. As $\sigma(\mathbf{v},\mathbf{e}_1) = 0$ and $\sigma(\mathbf{v},\mathbf{e}_2) = 0$ we have $v_1 = 0$ and $v_2 = 0$. Furthermore we have to show that $\sigma(\mathbf{v},\mathbf{e}_3+\mathbf{e}_4) = \sigma(\mathbf{v},\mathbf{e}_3)+\sigma(\mathbf{v},\mathbf{e}_4) = 0$. This implies $v_3 = v_4$. Thus we have $U^\perp = \{\lambda(\mathbf{e}_3+\mathbf{e}_4) : \lambda\in\mathbb{R}\} \subseteq U$ and $U + U^\perp \neq \mathbb{R}^4$.

d) There are three possibilities for signatures of $\sigma|_U$. We characterise them by their position w.r.t. the cone defined by the null set:

1) $(+,+,+)$: $\sigma|_U$ is non-degenerate and positive definite; U intersects the null cone only in $\{\mathbf{0}\}$, e.g. $U = \text{span}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$;
2) $(+,+,-)$: $\sigma|_U$ is non-degenerate but not positive definite; U intersects the null cone in a two dimensional cone, e.g. $U = \text{span}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_4)$;
3) $(+,+,0)$: $\sigma|_U$ is degenerate and positive definite on a two dimensional subspace of U, U is spanned by three vectors, two of them lie in the exterior of the cone and one lies on the cone; e.g. $U = \text{span}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3+\mathbf{e}_4)$;

Why are there no other signatures? Let $\mathbf{v}\neq\mathbf{0}$ with $\sigma(\mathbf{v},\mathbf{v}) = 0$. W.l.o.g. (after applying an σ-isometry according to (b)) we have $\mathbf{v} = \mathbf{e}_1+\mathbf{e}_4$. We look at an $\mathbf{w}\in\mathbf{v}^\perp$: $\mathbf{w} = \sum\lambda_i\mathbf{e}_i$. Then $\sigma(\mathbf{v},\mathbf{w}) = 0$ implies $\lambda_1 = \lambda_4$. So for $\sigma(\mathbf{w},\mathbf{w})$ we get: $\sigma(\mathbf{w},\mathbf{w}) = \lambda_2^2+\lambda_3^2 \geqslant 0$. This is only equal to 0, if λ_2 and λ_3 are equal to 0. This means that $\mathbf{w}$ is an scalar multiple of $\mathbf{v}$, and they are not linearly independent. This implies that besides an 0 the signature can have only positive entries. We still have the possibility of more than one negative entry. Let $\mathbf{v}$ be a vector with $\sigma(\mathbf{v},\mathbf{v}) = -1$. As before we can choose $\mathbf{v} = \mathbf{e}_4$. For $\mathbf{w}\in\mathbf{v}^\perp\setminus\{\mathbf{0}\}$ we have $\mathbf{w} = \lambda_1\mathbf{e}_1+\lambda_2\mathbf{e}_2+\lambda_3\mathbf{e}_3$. Therefore we get $\sigma(\mathbf{w},\mathbf{w}) = \lambda_1^2+\lambda_2^2+\lambda_3^2 > 0$. This implies that besides an negative entry only positive entries can occur in our signature.

Exercise 3 (A rotated ellipse)

Let X be the ellipse in $\mathbb{R}^2$ obtained by rotating the standard ellipse $\frac{x^2}{4}+y^2 = 1$ through the angle $-\frac{\pi}{6}$ and translating it so that its center is at the point $(1,-1)$. Find an equation for X.

Solution:

Rotating the ellipse through the angle $-\frac{\pi}{6}$ corresponds to rotating the coordinate axes by $\frac{\pi}{6}$. The corresponding rotation matrix is $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$. Therefore the change of variables $u = \frac{\sqrt{3}}{2}x+\frac{1}{2}y$, $v = -\frac{1}{2}x+\frac{\sqrt{3}}{2}y$ has the desired effect. We find $x = \frac{\sqrt{3}}{2}u-\frac{1}{2}v$ and $y = \frac{1}{2}u+\frac{\sqrt{3}}{2}v$. Substituting these into the equation $\frac{x^2}{4}+y^2 = 1$ yields $\frac{7}{16}u^2+\frac{13}{16}v^2+\frac{3\sqrt{3}}{8}uv = 1$. Finally, translating the ellipse so that its center is at $(1,1)$ yields $\frac{7}{16}(u-1)^2+\frac{13}{16}(v+1)^2+\frac{3\sqrt{3}}{8}(u-1)(v+1) = 1$. Multiplying this out yields

$$\frac{7}{16}u^2+\frac{13}{16}v^2+\frac{3\sqrt{3}}{8}uv+\frac{3\sqrt{3}-7}{8}u+\frac{-3\sqrt{3}+13}{8}v = \frac{3\sqrt{3}-2}{8}.$$

As in the lecture notes, this equation can be rewritten more succinctly in the form $\mathbf{u}^tA\mathbf{u}+\mathbf{b}^t\mathbf{u}+c = 0$, where $A = \begin{pmatrix} 7/16 & 3\sqrt{3}/16 \\ 3\sqrt{3}/16 & 13/16 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \frac{3\sqrt{3}-7}{8} \\ \frac{-3\sqrt{3}+13}{8} \end{pmatrix}$, and $c = -\frac{3\sqrt{3}-2}{8}$.

Exercise 4 (Projection onto a plane)

Let A be the affine plane in the euclidean space $(\mathbb{R}^3, \langle , \rangle)$ given by $x + 2y + 2z = 9$.

(a) Find an orthonormal basis for the 2-dimensional linear subspace $U \subseteq \mathbb{R}^3$ which is parallel to A.

(b) Extend this basis to an orthonormal basis B for $\mathbb{R}^3$.

(c) Write down the matrix which represents the orthogonal projection φ onto U in terms of the standard basis $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$.

(d) Let P be the point $(1, 2, -1)$. Find the shortest distance from P to A.

Solution:

a) U is precisely the subspace which is perpendicular to the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. To obtain an orthonormal basis for U, we apply the Gram-Schmidt process to the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, obtaining $\mathbf{u}_1 = \frac{4}{3\sqrt{2}} \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

b) We rescale the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ so that it has length 1, obtaining $\mathbf{v} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. By construction, $B = (\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2)$ is an orthonormal basis for $\mathbb{R}^3$ with the deisred properties.

c) The projection map φ has matrix $[\![\varphi]\!]_B^B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with respect to B. Since $[\![\mathrm{id}]\!]_E^B$ is the orthogonal matrix whose colums are $\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2$, we see that

$$[\![\varphi]\!]_E^E = [\![\mathrm{id}]\!]_E^B [\![\varphi]\!]_B^B [\![\mathrm{id}]\!]_B^E = \begin{pmatrix} 8/9 & -2/9 & -2/9 \\ -2/9 & 5/9 & -4/9 \\ -2/9 & -4/9 & 5/9 \end{pmatrix}.$$

d) The subspace U is just the plane passing through the origin given by $x + 2y + 2z = 0$. The point $Q = (9, 0, 0)$ clearly lies on A, so A is just the affine plane obtained by translating U by the vector $\begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$. In other words, (x, y, z) lies on A if and only if $(x - 9, y, z)$ lies on U. The distance from $P = (1, 2, -1)$ to A is therefore the same as the distance from $P' = (-8, 2, -1)$ to U. This is just the length of the vector $\begin{pmatrix} -8 \\ 2 \\ -1 \end{pmatrix} - [\![\varphi]\!]_E^E \begin{pmatrix} -8 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -4/3 \\ -4/3 \end{pmatrix}$, which is 2.