Linear Algebra II
Exercise Sheet no. 13

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Exercise 1 (Warm-up: symmetries of quadrics)
Consider the quadric

$$\mathbb{X} = \{\mathbf{v} \in \mathbb{R}^n : Q(\mathbf{v}) = c\},$$

where Q is a quadratic form over $\mathbb{R}^n$, $c \in \mathbb{R}$. Show that $\mathbb{X}$ is invariant under the following linear isometries of $\mathbb{R}^n$:

(a) $-\mathrm{id} : \mathbf{x} \mapsto -\mathbf{x}$ (central symmetry);

(b) reflection in the hyperplanes orthogonal to a principal axis (i.e., hyperplanes spanned by any $n-1$ basis vectors from an orthonormal basis that diagonalises Q and the associated σ);

(c) rotations in planes spanned by two principal axes w.r.t. which Q has the same "eigenvalues", i.e., by basis vectors $\mathbf{b}, \mathbf{b}'$ from an orthonormal basis that diagonalises Q such that $Q(\mathbf{b}) = Q(\mathbf{b}')$.

Solution:

a) $\mathbb{X}$ is invariant under central symmetry, since $Q(-\mathbf{v}) = (-1)^2 Q(\mathbf{v}) = Q(\mathbf{v})$.

b) Let $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be an orthonormal basis that diagonalises Q and φ be the reflection that maps $\mathbf{b}_1$ to $-\mathbf{b}_1$ and $\mathbf{b}_i$ to $\mathbf{b}_i$ for $i > 1$. Let $\mathbf{v} = \sum_i x_i \mathbf{b}_i \in \mathbb{X}$. Then for suitable $\lambda_i \in \mathbb{R}$,

$$Q(\varphi(\mathbf{v})) = Q(-x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_n\mathbf{b}_n) = \lambda_1(-x_1)^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 = Q(\mathbf{v}).$$

c) Let $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be an orthonormal basis that diagonalises Q and satisfies $Q(\mathbf{b}_1) = Q(\mathbf{b}_2)$, and let φ be the rotation that maps $\mathbf{b}_1$ to $\cos(\alpha)\mathbf{b}_1 - \sin(\alpha)\mathbf{b}_2$, $\mathbf{b}_2$ to $\sin(\alpha)\mathbf{b}_1 + \cos(\alpha)\mathbf{b}_2$ and $\mathbf{b}_i$ to $\mathbf{b}_i$ for $i > 2$. Let $\mathbf{v} = \sum_i x_i \mathbf{b}_i \in \mathbb{X}$. Then for suitable $\lambda_i \in \mathbb{R}$ with $\lambda_1 = \lambda_2$:

$$\begin{aligned}
Q(\varphi(\mathbf{v})) &= Q\big((x_1\cos(\alpha) + x_2\sin(\alpha))\mathbf{b}_1 + (-x_1\sin(\alpha) + x_2\cos(\alpha))\mathbf{b}_2 + \\
&\quad + x_3\mathbf{b}_3 + \ldots + x_n\mathbf{b}_n\big) \\
&= \lambda_1(x_1\cos(\alpha) + x_2\sin(\alpha))^2 + \lambda_2(-x_1\sin(\alpha) + x_2\cos(\alpha))^2 + \\
&\quad + \lambda_3 x_3^2 + \ldots + \lambda_n x_n^2 \\
&= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \ldots + \lambda_n x_n^2 \\
&= Q(\mathbf{v}),
\end{aligned}$$

which proves the claim.

Exercise 2 (Canonical form of a quadric)
Let $\mathbb{X}$ be the set of all points $\mathbf{x} \in \mathbb{R}^3$ satisfying the following equation:

$$2x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 + 4x_1 - 3x_2 - x_3 = 0.$$

(a) Find a matrix A and a vector $\mathbf{b}$ such that the given equation can be written as

$$\mathbf{x}^t A \mathbf{x} + \mathbf{b}^t \mathbf{x} = 0.$$

(b) Find an affine transformation for $\mathbb{R}^3$ for which the given equation has the form

$$a(x_1'+c_1)^2+b(x_2'-c_2)^2+x_3'-c_3=0.$$

(c) Describe the set $\mathbb{X}$ geometrically.

Solution:

a) $A=\begin{pmatrix}2&0&0\\0&1&1\\0&1&1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}4\\-3\\-1\end{pmatrix}$.

b) The eigenvalues of A are $2,2,0$. $\begin{pmatrix}1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$ is a basis of normalized eigenvectors and let S be the basis transformation from the standard basis to our new basis. With $\mathbf{x}=S\mathbf{x}'$ we compute

$$\begin{aligned}0 &= (\mathbf{x}')^tS^tAS\mathbf{x}'+\mathbf{b}^tS\mathbf{x}'=(\mathbf{x}')^t\begin{pmatrix}2&0&0\\0&2&0\\0&0&0\end{pmatrix}x'+(4-2\sqrt{2}-\sqrt{2})\mathbf{x}'\\ &= 2(x_1')^2+2(x_2')^2+4x_1'-2\sqrt{2}x_2'-\sqrt{2}x_3'\\ &= 2(x_1'+1)^2-2+2(x_2'-\frac{1}{\sqrt{2}})^2-1-\sqrt{2}x_3'\\ &= 2(x_1'+1)^2+2(x_2'-\frac{1}{\sqrt{2}})^2-\sqrt{2}x_3'-3\end{aligned}$$

So we get the following equation:

$$-\sqrt{2}(x_1'+1)^2-\sqrt{2}(x_2'-\frac{1}{\sqrt{2}})^2+x_3+\frac{3}{\sqrt{2}}=0.$$

c) $\mathbb{X}$ is the surface obtained by rotating a parabola about the axis defined by $x_1'=-1$ and $x_2'=\frac{1}{\sqrt{2}}$.

Exercise 3 (Three-dimensional quadrics)

Various three-dimensional affine quadrics as for instance the single-sheet hyperboloid $\{(x,y,z): x^2+y^2-z^2=1\}$ and the saddle surface $\{(x,y,z): x^2-y^2=z\}$, can be seen to be different affine sections of the projective quadric

$$\mathbb{X}=\{[\mathbf{x}]: x_1x_2-x_3x_4=0\}. \qquad (*)$$

(a) Find the matrix representing $\mathbb{X}$ w.r.t. the standard basis.

(b) Diagonalise the quadratic form for $\mathbb{X}$.

(c) Use $(*)$ to find an affine hyperplane whose intersection with $\mathbb{X}$ is a saddle surface and use (b) to find a plane whose intersection with $\mathbb{X}$ is a single-sheet hyperboloid.

(d) Give a homogeneous equation for another projective quadric $\mathbb{X}'$ such that there exist two affine hyperplanes whose intersections with $\mathbb{X}'$ are a double-sheet hyperboloid and an ellipsoid, respectively.

Solution:

a) $A=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}$

b) The characteristic polynomial of A is $(\lambda^2-1)^2$. So we have eigenvalues 1 and -1, each of them twice. Therefore we can choose a suitable orthonormal basis such that $\mathbb{X}$ is defined by the equation $y_1^2+y_2^2-y_3^2-y_4^2=0$.

c) We choose the hyperplane $x_1=1$ and get $x_2-x_3x_4=0$. For new coordinates $z_1=\frac{1}{2}(x_3+x_4), z_2=\frac{1}{2}(x_4-x_3)$ instead of x_3 and x_4 we get $x_2-z_1^2+z_2^2=0 \Leftrightarrow x_2=z_2^2-z_1^2$. This intersection is a hyperbolic paraboloid (saddle surface). In the equation obtained in (b) we set $y_1=1$ and get $y_3^2+y_4^2-y_2^2=1$. This equation describes a single-sheet hyperboloid.

d) We take the equation $x_1^2+x_2^2+x_3^2-x_4^2=0$ that corresponds to the signature $(1,1,1,-1)$. Intersecting $\mathbb{X}'$ with the affine hyperplane $x_1=1$, we obtain the quadric $x_2^2+x_3^2-x_4^2=-1$. This is a two-sheet hyperboloid. Intersecting with $x_4=1$ we get the quadric $x_1^2+x_2^2+x_3^2=1$, which defines an ellipsoid.

Exercise 4 (Snake on a plane)

A snake wants to buy a blanket. The snake's length is one unit, and we assume that it can bend any which way, but may always be described by a smooth curve of length 1. The snake wants to make sure that it can cover itself with the blanket no matter in which shape it wants to lie down. Obviously a round blanket of diameter 1 is good enough (why?). A clever shop assistant points out that a half disc of diameter 1 should also suffice.

Prove that this is right: any length 1 curve in $\mathbb{R}^2$ can be covered by a half disc of diameter 1.

Hint: Consider the end points and the mid point of the curve, and use the fact that the whole of the snake lies within the union of the two ellipses formed by the mid point with either end point as foci and with length 1/2 for the sum of distances from the foci. It now suffices to show that any two such ellipses are contained within a half disc of radius 1/2 whose straight boundary is a common tangent to the two ellipses, and whose centre point is the orthogonal projection of the shared focus point onto this tangent. (See Exercise E12.5.)

Solution:

Let S be a snake lying in the plane $\mathbb{R}^2$. Let H and T be the two ends of the snake, and P be the midpoint of the snake. (These stand for head, tail and pancreas.) Let E_H be the region bounded by the ellipse consisting of the points M such that $d(H,M)+d(M,P)=\frac{1}{2}$. By definition of E_H and by the triangle inequality, the fore-half of the snake must lie withing E_H. Let us define E_T similarly, so that $S \subseteq E_H \cup E_T$. Let L be a tangent to both E_H and E_T (we assume the existence of such a line), so that both ellipses lie fully on the same side of the line. Let O be the orthogonal projection of P onto L. By Exercise E12.5.c, the smallest circle including, e.g., E_H intersects L at O. The diameter of this circle is $\frac{1}{2}$ (like the big diameter of the ellipse), so every point within the circle is at distance at most $\frac{1}{2}$ from O. Therefore every point of the fore-half of the snake is at distance at most $\frac{1}{2}$ from O. Since this holds also for the other half-snake, we are done.