Linear Algebra II
Exercise Sheet no. 12

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Exercise 1 (Warm-up: diagonalisability of bilinear forms)
Let the bilinear forms σ_1 and σ_2 on $\mathbb{R}^3$ be defined by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

with respect to the standard basis of $\mathbb{R}^3$.

(a) Determine an orthonormal basis of $\mathbb{R}^3$ with respect to which the matrix of σ_1 is diagonal.

(b) Show that every eigenvector of A_2 is also an eigenvector of A_1.

(c) Determine an orthonormal basis of $\mathbb{R}^3$ with respect to which the matrix of σ_2 is diagonal, and deduce the eigenvalues of A_2 without computing the characteristic polynomial.

Solution:

a) We determine an orthonormal basis B_1 of eigenvectors for each of the matrices A_1. These eigenvectors form the columns of a transformation matrix C_1. Then $[\![\sigma_i]\!]^{B_1} = C_1{}^t A_1 C_1$ is the matrix of σ_1 in this basis and, by Proposition 3.2.9 (A_1 is symmetric), $[\![\sigma_1]\!]^{B_1}$ is diagonal.

The eigenvalues of A_1 are given by

$$0 = \det(A_1 - \lambda E) = \lambda^2(2 - \lambda),$$

hence they are 0, with algebraic and geometric multiplicity two, and 2. We determine now the eigenvectors:

For $\lambda = 0$:

$$\ker(A_1) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2), \quad \text{where } \mathbf{v}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

For $\lambda = 2$:

$$\ker(A_1 - 2E) = \operatorname{span}(\mathbf{v}_3),$$

where $\mathbf{v}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Then $B_1 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $[\![\sigma_1]\!]^{B_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b) Assume that (x, y, z) is an eigenvector of A_2 with eigenvalue λ. Then $2x + z = \lambda x$ and $x + 2z = \lambda z$, so $(3 - \lambda)(x + z) = 0$. If $\lambda = 3$ then $x = z$ since $2x + z = \lambda x$, and if $\lambda \neq 3$ then $x = -z$. In both cases (x, y, z) is an eigenvector of A_1.

c) The same basis $B_1 = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ as in question (a) works due to question (b). So $[\![\sigma_2]\!]^{B_1} = C_1{}^t A_2 C_1$ is diagonal and its entries are the eigenvalues, namely 1 (multiplicity two) and 3.

Exercise 2 (Quadratic forms)

Which of the following are quadratic forms? Determine in each case the corresponding symmetric bilinear forms:

(a) $Q_1 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x.$

(b) $Q_2 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 0.$

(c) $Q_3 : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto -3x_1^2 - x_2^2 - x_2 x_1.$

(d) $Q_4 : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto (\sqrt{x_1} + \sqrt{x_2})^4.$

(e) $Q_5 : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto ||\mathbf{x}||^2.$

Solution:

a) This is not a quadratic form. For instance, $Q_1(2) = 2 \neq 4 = 2^2 Q_1(1)$.

b) This is a quadratic form and $\sigma_2(x, y) = 0$

c) This is a quadratic form and $\sigma_3(x, y) = -3x_1 y_1 - x_2 y_2 - x_1 y_2 - x_2 y_1$.

d) Let $\sigma_4(\mathbf{x}, \mathbf{y}) := \frac{1}{2}\left(Q_4(\mathbf{x} + \mathbf{y}) - Q_4(\mathbf{x}) - Q_4(\mathbf{y})\right)$
So $\sigma_4((x_1, x_2), (y_1, y_2)) = \frac{1}{2}\left((\sqrt{x_1 + y_1} + \sqrt{x_2 + y_2})^2 4(\sqrt{x_1} + \sqrt{x_2})^4 - (\sqrt{y_1} + \sqrt{y_2})^4\right)$.

By observation 3.3.2 in the notes, Q_4 is a quadratic form iff this σ_4 is bilinear. We next show that σ_4 is not bilinear in order to show that Q_4 is not a quadratic form. By definition we have $2\sigma_4((1,0),(0,1)) = 2^4 - 2$ and $2\sigma_4((0,1),(0,1)) = 2$ and $2\sigma_4((1,1),(0,1)) = (1 + \sqrt{2})^4 - 2^2 - 1$, the latter being irrational. (Recall that $(a+b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$ and that $\sqrt{2}$ is irrational.) Therefore $\sigma_4((1,1),(0,1)) \neq \sigma_4((1,0),(0,1)) + \sigma_4((0,1),(0,1))$, and Q_4 is not a quadratic form.

e) This is a quadratic form since $Q_5(x_1, x_2) = x_1^2 + x_2^2$ is induced by the standard scalar product. (That is, $\sigma_5((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 y_2$.)

Exercise 3 (Transformation of quadratic forms)

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the endomorphism represented w.r.t. the standard basis of $\mathbb{R}^2$ by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\mathbb{S}^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}^t \mathbf{x} = 1\}.$$

be the unit circle in $\mathbb{R}^2$.

(a) Describe the image of the unit circle $\mathbb{S}^1$ under φ, $\varphi[\mathbb{S}^1] \subseteq \mathbb{R}^2$, by a corresponding equation.

(b) Determine a symmetric bilinear form σ such that

$$\varphi[\mathbb{S}^1] = \{\mathbf{x} \in \mathbb{R}^2 : \sigma(\mathbf{x}, \mathbf{x}) = 1\}.$$

(c) Find the symmetry axes of $\varphi[\mathbb{S}^1]$.
Hint: apply Theorem 3.2.5 to σ.

Solution:

Note that A is regular and $A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

a) Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$. It follows that $\mathbf{x} \in \varphi[\mathbb{S}^1]$ iff $\varphi^{-1}(\mathbf{x}) \in \mathbb{S}^1$ iff

$$\begin{aligned} 1 &= (\varphi^{-1}(\mathbf{x}))^t \varphi^{-1}(\mathbf{x}) = (A^{-1}\mathbf{x})^t A^{-1}\mathbf{x} = \mathbf{x}^t (A^{-1})^t A^{-1}\mathbf{x} = \mathbf{x}^t \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^t \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{x} \\ &= \mathbf{x}^t \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \mathbf{x}^t \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{x} \\ &= x_1^2 - 2x_1x_2 + 2x_2^2. \end{aligned}$$

Hence

$$\varphi[\mathbb{S}^1] = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 - 2x_1x_2 + 2x_2^2 = 1\} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}^t B \mathbf{x} = 1\},$$

where $C = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

b) Let σ be the symmetric bilinear form represented by C w.r.t. the standard basis of $\mathbb{R}^2$. Then

$$\varphi(\mathbb{S}^1) = \{\mathbf{x} \in \mathbb{R}^2 : \sigma(\mathbf{x}, \mathbf{x}) = 1\}.$$

c) The characteristic polynomial of C is

$$p_C = \det(C - XE) = (1 - X)(2 - X) - 1 = \left(X - \frac{3 + \sqrt{5}}{2}\right)\left(X - \frac{3 - \sqrt{5}}{2}\right),$$

The eigenvalues of C are then $\lambda_1 = \frac{3-\sqrt{5}}{2}, \lambda_2 = \frac{3+\sqrt{5}}{2}$ and the signature of σ is $(+,+)$. Therefore $\varphi[\mathbb{S}^1]$ is an ellipse.

An orthonormal basis of eigenvectors is $B = (\mathbf{v}_1, \mathbf{v}_2)$, where

$$\begin{aligned} \mathbf{v}_1 &= \frac{1}{\sqrt{1 + (1 - \lambda_1)^2}} \begin{pmatrix} 1 \\ 1 - \lambda_1 \end{pmatrix} = \frac{\sqrt{2}}{\sqrt{5 - \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix} \quad \text{und} \\ \mathbf{v}_2 &= \frac{1}{\sqrt{1 + (1 - \lambda_2)^2}} \begin{pmatrix} 1 \\ 1 - \lambda_2 \end{pmatrix} = \frac{\sqrt{2}}{\sqrt{5 + \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix} \end{aligned}$$

The symmetry axes of the ellipse $\varphi[\mathbb{S}^1]$ are the principal axes of σ, hence the eigenspaces $V_{\lambda_1} = \operatorname{span}(\mathbf{v}_1)$ and $V_{\lambda_2} = \operatorname{span}(\mathbf{v}_2)$.

Exercise 4 (Quadratic forms)

Determine the principal axes and the signature of the following quadratic forms

(a) $Q_1(\mathbf{x}) = -11x_1^2 - 16x_1x_2 + x_2^2$,

(b) $Q_2(\mathbf{x}) = 9x_1^2 - 4x_1x_2 + 6x_2^2$,

(c) $Q_3(\mathbf{x}) = 4x_1^2 - 12x_1x_2 + 9x_2^2$,

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

Solution:

a) $A_1 := [\![Q_1]\!]^B = \begin{pmatrix} -11 & -8 \\ -8 & 1 \end{pmatrix}$.

Characteristic polynomial: $(X - 5)(X + 15)$

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = -15$.

Eigenvectors for these eigenvalues: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Signature: $(+, -)$

Principal axes: $V_5 = \operatorname{span}((1, -2)^t)$ and $V_{-15} = \operatorname{span}((2, 1)^t)$.

b) $A_2 := [\![Q_2]\!]^B = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$.

Characteristic polynomial: $(X-5)(X-10)$

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 10$.

Eigenvectors for these eigenvalues: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Signature: $(+,+)$

Principal axes: $V_5 = \text{span}((1,-2)^t)$ and $V_{10} = \text{span}((2,1)^t)$.

c) $A_3 := [\![Q_3]\!]^B = \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}$.

Characteristic polynomial: $X(X-3)$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 3$.

Eigenvectors for these eigenvalues: $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Signature: $(+,0)$

Principal axes: $V_0 = \text{span}((3,2)^t)$ and $V_3 = \text{span}((-2,3)^t)$.

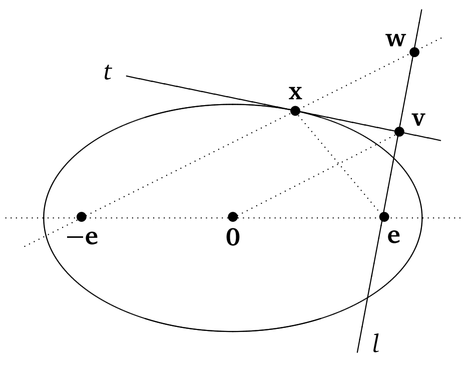
Exercise 5 (Geometric properties of the ellipse in euclidean geometry)

Let $0 \leqslant e < 1$ and consider the points given by the vectors $\mathbf{e} = (e,0)$ and $-\mathbf{e} = (-e,0)$ in the real plane $\mathbb{R}^2$. Let the set $\mathbb{X}_e \subseteq \mathbb{R}^2$ be defined as the set of all those $\mathbf{x} \in \mathbb{R}^2$ for which $d(\mathbf{x},\mathbf{e}) + d(\mathbf{x},-\mathbf{e}) = 2$.

(a) Show that $\mathbb{X}_e$ is an *ellipse* defined by a quadratic equation of the form $\alpha x^2 + \beta y^2 = 1$ for suitable $\alpha, \beta > 0$. Determine α and β in terms of e. Draw $\mathbb{X}_e$ for $e = 0, 1/2, 1, 1/\sqrt{2}$.

(b) From (i) find a representation of $\mathbb{X}_e$ as the image of the unit circle under a rescaling in the y-direction. Use this rescaling and the fact that linear transformations preserve the property that a line is a tangent to a curve in order to determine the equation of the tangent to the ellipse $\mathbb{X}_e$ in a point $\mathbf{x} = (x,y) \in \mathbb{X}_e$. Show that lines from $\mathbf{e}$ and $-\mathbf{e}$ through $\mathbf{x}$ form the same angle with the tangent at $\mathbf{x}$. [This explains the rôle of the points $\mathbf{e}$ and $-\mathbf{e}$ as the *foci* of the ellipse: light shining from $\mathbf{e}$ is focussed in $-\mathbf{e}$ after reflection in $\mathbb{X}_e$.]

(c) Show by elementary geometric means that $\mathbb{X}_e$ also has the following geometric property. Let t be the tangent to $\mathbb{X}_e$ in a point $\mathbf{x} \in \mathbb{X}_e$ and l the line through $\mathbf{e}$ perpendicular to t. Then the point of intersection $\mathbf{v}$ between l and t lies on the unit circle.
Hint: Consider the triangles $(\mathbf{x},\mathbf{v},\mathbf{w})$ and $(\mathbf{x},\mathbf{v},\mathbf{e})$ in the sketch below, where $\mathbf{v}$ marks the point where l intersects t, and $\mathbf{w}$ where it intersects the line through $-\mathbf{e}$ and $\mathbf{x}$. Use (ii) to argue that these triangles are congruent.



Solution:

a) When $\mathbf{x} = (x,y)$, then

$$\begin{aligned}
& d(\mathbf{x},-\mathbf{e}) + d(\mathbf{x},\mathbf{e}) = 2 \\
\Leftrightarrow \quad & \sqrt{(x+e)^2+y^2} + \sqrt{(x-e)^2+y^2} = 2 \\
\Leftrightarrow \quad & (x+e)^2+y^2+(x-e)^2+y^2+2\sqrt{(x+e)^2+y^2}\sqrt{(x-e)^2+y^2} = 4 \\
\Leftrightarrow \quad & 2-x^2-y^2-e^2 = \sqrt{(x+e)^2+y^2}\sqrt{(x-e)^2+y^2} \\
\Leftrightarrow^* \quad & (2-x^2-y^2-e^2)^2 = ((x+e)^2+y^2)((x-e)^2+y^2) \\
\Leftrightarrow \quad & (1-e^2)x^2+y^2 = 1-e^2 \\
\Leftrightarrow \quad & x^2 + \tfrac{1}{1-e^2}y^2 = 1.
\end{aligned}$$

(For the $\Leftarrow$-direction at (*), observe that $(1-e^2)x^2 + y^2 = 1 - e^2$ implies that $y^2 + e^2 \leqslant 1$, and $x^2 + \frac{1}{1-e^2}y^2 = 1$ implies that $x^2 \leqslant 1$, so that $2 - x^2 - y^2 - e^2 \geqslant 0$.)

b) The ellipse $\mathbb{X}_e$ is the image of the unit circle under the linear map $\varphi_e : (x, y) \mapsto (x, \sqrt{(1-e^2)}y)$.

Let $\mathbf{x} = (x, y)$ be a point on the ellipse $\mathbb{X}_e$, and assume (without loss of generality) that $x, y > 0$, and let $\mathbf{x}' = (x, y')$ be the point on the unit circle such that $\varphi_e(\mathbf{x}') = \mathbf{x}$. The tangent to the unit circle in $\mathbf{x}'$ passes through $\mathbf{x}'$ and the point $(1/x, 0)$ (check!). Therefore the tangent through $\mathbf{x}$ to the ellipse $\mathbb{X}_e$ is the line through $\mathbf{x} = \varphi_e(\mathbf{x}')$ and $(1/x, 0) = \varphi_e(1/x, 0)$, which in parameter form is:

$$t = \{x + \lambda(x^2 - 1, xy) : \lambda \in \mathbb{R}\}.$$

Therefore the vector $\mathbf{b} = (xy, 1 - x^2)$ is orthogonal to the tangent t at $\mathbf{x} = (x, y)$.

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the difference vectors $\mathbf{a}_1 = \mathbf{x} - (-\mathbf{e})$ and $\mathbf{a}_2 = \mathbf{x} - \mathbf{e}$. Equality of the angles of $\mathbf{a}_1$ and $\mathbf{a}_2$ with the tangent is then equivalent to

$$\frac{\langle \mathbf{b}, \mathbf{a}_1 \rangle}{\|\mathbf{a}_1\|} = \frac{\langle \mathbf{b}, \mathbf{a}_2 \rangle}{\|\mathbf{a}_2\|}.$$

As both scalar products are positive for $x, y > 0$, this is equivalent to

$$\frac{\langle \mathbf{b}, \mathbf{a}_1 \rangle^2}{\|\mathbf{a}_1\|^2} = \frac{\langle \mathbf{b}, \mathbf{a}_2 \rangle^2}{\|\mathbf{a}_2\|^2},$$

which is equivalent to

$$\frac{(xy(x+e) + (1-x^2)y)^2}{(x+e)^2 + y^2} = \frac{(xy(x-e) + (1-x^2)y)^2}{(x-e)^2 + y^2},$$

which in turn, after some lengthy but standard calculations, can be seen to be equivalent to the defining equation $x^2(1-e^2) + y^2 = 1 - e^2$.

c) Congruence of the triangles $(\mathbf{x}, \mathbf{v}, \mathbf{w})$ and $(\mathbf{x}, \mathbf{v}, \mathbf{e})$ follows from the fact that (a) they share the common side $\mathbf{xv}$, (b) t is perpendicular to l, so the angles $\angle\mathbf{xvw}$ and $\angle\mathbf{xve}$ are both straight, and (c) the angles $\angle\mathbf{vxw}$ and $\angle\mathbf{vxe}$ are equal by (ii).

Therefore $d(-\mathbf{e}, \mathbf{w}) = d(-\mathbf{e}, \mathbf{x}) + d(\mathbf{x}, \mathbf{e}) = 2$, from which it follows that $\|\mathbf{v}\| = d(\mathbf{0}, \mathbf{v}) = \frac{1}{2}d(-\mathbf{e}, \mathbf{w}) = 1$.