Linear Algebra II
Exercise Sheet no. 10

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Exercise 1 (Warm-up: self-adjoint maps)
Let V be a finite-dimensional unitary space and $\varphi \in \mathrm{Hom}(V, V)$. Show that the following are equivalent:

(a) φ is self-adjoint.

(b) $\langle \mathbf{v}, \varphi(\mathbf{v})\rangle \in \mathbb{R}$ for all $\mathbf{v} \in V$.

Hint: Consider $\langle \mathbf{v}+\mathbf{w}, \varphi(\mathbf{v}+\mathbf{w})\rangle$ and $\langle \mathbf{v}+i\mathbf{w}, \varphi(\mathbf{v}+i\mathbf{w})\rangle$ for the implication (b)⇒(a).

Solution:
(a)⇒(b). Assume that φ is self-adjoint. Then for all $\mathbf{v} \in V$,

$$\langle T\mathbf{v}, \mathbf{v}\rangle - \overline{\langle T\mathbf{v}, \mathbf{v}\rangle} \quad = \quad \langle T\mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{v}, T\mathbf{v}\rangle = 0,$$

hence $\langle T\mathbf{v}, \mathbf{v}\rangle \in \mathbb{R}$.
(b) ⇒ (a). If $\langle \mathbf{v}, \varphi(\mathbf{v})\rangle \in \mathbb{R}$ for all $\mathbf{v} \in V$, then for any pair of vectors $\mathbf{v}, \mathbf{w} \in V$:

$$\left.\begin{array}{rl} \langle \mathbf{v}+\mathbf{w}, \varphi(\mathbf{v}+\mathbf{w})\rangle - \langle \mathbf{v}, \varphi(\mathbf{v})\rangle - \langle \mathbf{w}, \varphi(\mathbf{w})\rangle = & \langle \mathbf{w}, \varphi(\mathbf{v})\rangle + \langle \mathbf{v}, \varphi(\mathbf{w})\rangle \\ \langle \mathbf{v}+i\mathbf{w}, \varphi(\mathbf{v}+i\mathbf{w})\rangle - \langle \mathbf{v}, \varphi(\mathbf{v})\rangle - \langle \mathbf{w}, \varphi(\mathbf{w})\rangle = & i\big(\langle \mathbf{w}, \varphi(\mathbf{v})\rangle - \langle \mathbf{v}, \varphi(\mathbf{w})\rangle\big) \end{array}\right\} \in \mathbb{R},$$

whence $\mathrm{im}(\langle \mathbf{w}, \varphi(\mathbf{v})\rangle) = -\mathrm{im}(\langle \mathbf{v}, \varphi(\mathbf{w})\rangle)$ and $\mathrm{Re}(\langle \mathbf{w}, \varphi(\mathbf{v})\rangle) = \mathrm{Re}(\langle \mathbf{v}, \varphi(\mathbf{w})\rangle)$. It follows that

$$\langle \mathbf{v}, \varphi(\mathbf{w})\rangle = \overline{\langle \mathbf{w}, \varphi(\mathbf{v})\rangle} = \langle \varphi(\mathbf{v}), \mathbf{w}\rangle = \langle \mathbf{v}, \varphi^{+}(\mathbf{w})\rangle$$

for all $\mathbf{v}, \mathbf{w} \in V$, hence $\varphi = \varphi^{+}$.

Exercise 2 (Eigenvalues)
Let V be a finite dimensional vector space and φ, ψ be endomorphisms of V.
Prove that λ is an eigenvalue of $\varphi \circ \psi$ if and only if it is an eigenvalue of $\psi \circ \varphi$.
Hint: It may help to distinguish cases according to whether $\lambda \neq 0$ or $\lambda = 0$.
Extra: Can you give a counterexample in case V is infinite dimensional?

Solution:
Let λ be an eigenvalue of $\varphi \circ \psi$. We have two cases:

a) $\lambda \neq 0$:
There is an eigenvector $\mathbf{v} \neq 0$ with $(\varphi \circ \psi)(\mathbf{v}) = \lambda \mathbf{v}$. This yields

$$(\psi \circ \varphi)(\psi(\mathbf{v})) = \big((\psi \circ \varphi) \circ \psi\big)(\mathbf{v}) = \big(\psi \circ (\varphi \circ \psi)\big)(\mathbf{v}) = \psi\big((\varphi \circ \psi)(\mathbf{v})\big) = \psi(\lambda \mathbf{v}) = \lambda \psi(\mathbf{v}).$$

Because $\lambda \neq 0$ and $\mathbf{v} \neq \mathbf{0}$ we know that $\psi(\mathbf{v}) \neq \mathbf{0}$ as well (otherwise $\lambda \mathbf{v} = (\varphi \circ \psi)(\mathbf{v}) = \varphi(\psi(\mathbf{v})) = \varphi(\mathbf{0}) = \mathbf{0}$, a contradiction). Thus $\psi(\mathbf{v})$ is an eigenvector of $\psi \circ \varphi$ with eigenvalue λ.

b) $\lambda = 0$:
As V is finite dimensional, the maps φ and ψ can be described with respect to a basis of V by matrices A resp. B. Since $\lambda = 0$ is a root of $\det(AB - \lambda E) = 0$, $\det(AB) = 0$. This implies that $\det(BA) = \det(B)\det(A) = \det(A)\det(B) = \det(AB) = 0$, and therefore $\lambda = 0$ is a solution of $\det(BA - \lambda E) = 0$, i.e. λ is an eigenvalue of $\psi \circ \varphi$.

Extra: Let V be an infinite dimensional euclidean space and U be a proper subspace of V such that there exists an isomorphism $\varphi : V \to U$. Take $\psi := \pi_U$.

Exercise 3 (Self-adjoint and unitary maps)
Let V be a finite-dimensional unitary space and $\varphi \in \mathrm{Hom}(V,V)$ be a normal endomorphism. Show the following.

(a) φ is self-adjoint if and only if all the eigenvalues of φ are real.

(b) φ is unitary if and only if all the eigenvalues of φ have absolute value 1.

Solution:

a) $\Rightarrow$ By Proposition 2.4.5.

$\Leftarrow$ Assume conversely that all the eigenvalues are real. Since φ is normal, there exists an orthonormal basis of eigenvectors of φ, which are at the same time eigenvectors of φ^+. With respect to this basis, both φ and φ^+ are represented by diagonal matrices. These matrices are identical, since the eigenvalues of φ^+ are the conjugates of the eigenvalues of φ, and these are real.

b) $\Rightarrow$ If φ is unitary and λ is an eigenvalue of φ with corresponding eigenvector $\mathbf{v}$, then

$$\|\mathbf{v}\| = \|\varphi(\mathbf{v})\| = \|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|,$$

hence $|\lambda| = 1$.

$\Leftarrow$ Assume conversely that $|\lambda| = 1$ for every eigenvalue λ of φ. Since φ and, hence, φ^+ are normal, by Exercise (T 10.2), they have a common orthonormal basis of eigenvectors with respect to which they are represented by diagonal matrices. Let us denote with D the diagonal matrix representing φ and with $D^+ = \overline{D}$ the one representing φ^+.

Let $i \in \{1,\dots,\dim(V)\}$ be arbitrary. The i-th row of D has the form $(0,\dots,0,\lambda,0,\dots,0)$, with an eigenvalue $\lambda \in \mathbb{C}$. Correspondingly, the i-th column of D^+ is $(0,\dots,0,\overline{\lambda},0,\dots,0)^t$. It follows that the product DD^+ is a diagonal matrix with $|\lambda\overline{\lambda}| = 1$ as the i-th diagonal entry. Since i was arbitrary, we get that $DD^+ = E$, hence φ is unitary.

Exercise 4 (Simultaneous diagonalization)
Let V be a finite dimensional unitary space and $\varphi_1,\dots,\varphi_m$ normal endomorphisms of V that pairwise commute, that is $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for all $i,j \in \{1,\dots,m\}$.

Prove that there exists an orthonormal basis $B = (\mathbf{b}_1,\dots,\mathbf{b}_n)$ of V consisting of *simultaneous eigenvectors*, that is there are complex numbers λ_{ij} for $i = 1,\dots,m$ and $j = 1,\dots,n$, such that

$$\varphi_i(\mathbf{v}_j) = \lambda_{ij}\mathbf{v}_j$$

for all i,j.

(a) Let λ be an eigenvalue of φ_1 and $V_\lambda(\varphi_1) = \{\mathbf{v} \in V \mid \varphi_1(\mathbf{v}) = \lambda\mathbf{v}\}$ the corresponding eigenspace. Prove that

$$\varphi_i(V_\lambda(\varphi_1)) \subseteq V_\lambda(\varphi_1)$$

for all i.

(b) Let λ and μ be two different eigenvalues of φ_i. Show that the corresponding eigenspaces are orthogonal.

(c) Prove now the existence of a basis of V with the desired properties.
Hint: *Induction on m.*

Solution:

a) Let $v \in V_\lambda(\varphi_1)$. Since $\varphi_i \circ \varphi_1 = \varphi_1 \circ \varphi_i$, we get that

$$\varphi_1\big(\varphi_i(\mathbf{v})\big) = (\varphi_1 \circ \varphi_i)(\mathbf{v}) = (\varphi_i \circ \varphi_1)(\mathbf{v}) = \varphi_i\big(\varphi_1(\mathbf{v})\big) = \varphi_i(\lambda\mathbf{v}) = \lambda\varphi_i(\mathbf{v}),$$

therefore $\varphi_i(\mathbf{v}) \in V_\lambda(\varphi_1)$.

b) Let λ and μ be different eigenvalues of φ_i. Let now $\mathbf{v} \in V_\lambda(\varphi_i)$ and $\mathbf{w} \in V_\mu(\varphi_i)$ be arbitrary. Since φ_i is normal, we can apply Exercise (T 9.1) to get that

$$\mu\langle\mathbf{v},\mathbf{w}\rangle = \langle\mathbf{v},\mu\mathbf{w}\rangle = \langle\mathbf{v},\varphi_i(\mathbf{w})\rangle = \langle\varphi_i^+(\mathbf{v}),\mathbf{w}\rangle = \langle\overline{\lambda}\mathbf{v},\mathbf{w}\rangle = \lambda\langle\mathbf{v},\mathbf{w}\rangle.$$

As $\lambda \neq \mu$, this is only possible only when $\langle\mathbf{v},\mathbf{w}\rangle = 0$.

c) We prove the statement by induction on m. The case $m=1$ is clear, by Theorem 2.4.10.

The **Induction Step** (from $m-1$ to m) results from (a) and (b): By (a), the eigenspace $V_\lambda(\varphi_1)$ is an invariant subspace of φ_i for $i=2,\dots,m$. It follows that these endomorphisms can be restricted to $V_\lambda(\varphi_1)$. The restrictions are normal (with respect to the restriction of the scalar product to $V_\lambda(\varphi_1)$). By the induction hypothesis, there exists an orthonormal basis of $V_\lambda(\varphi_1)$ consisting of simultaneous eigenvectors of φ_i, $i=2,\dots,m$, which are obvious eigenvectors of φ_1 (with eigenvalue λ).

Since, by (b), the eigenspaces of φ_1 w.r.t other eigenvalues are orthogonal on $V_\lambda(\varphi_1)$, we obtain altogether an orthonormal basis of V with the desired properties.

Exercise 5 (Isometries and 'skew-rotations')

We consider the real plane $\mathbb{R}^2$ with the standard scalar product $\langle .,. \rangle$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that is represented by a rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with respect to some basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. We assume that $\theta \neq 0, \pi$.

Show that φ is an isometry if and only if B is almost an orthonormal basis in the sense that

$$\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = 0 \quad \text{and} \quad \langle \mathbf{b}_1, \mathbf{b}_1 \rangle = \langle \mathbf{b}_2, \mathbf{b}_2 \rangle .$$

(So we require the lengths of $\mathbf{b}_1$ and $\mathbf{b}_2$ only to be equal, not to be 1.)

Solution:

Let $G := [\![\langle .,. \rangle]\!]^B$ be the matrix for $\langle .,. \rangle$ with respect to the basis B. We have to prove that φ is an isometry if and only if

$$G = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = cE_2$$

for some $c \in \mathbb{R}$.

We start by showing that φ is an isometry if and only if the matrices G and A commute, i.e.,

$$GA = AG .$$

Note that A being orthogonal, we have $A^t = A^{-1}$. Hence, $GA = AG$ implies that

$$\begin{aligned} \langle \varphi(\mathbf{x}), \varphi(\mathbf{y}) \rangle &= (A[\![\mathbf{x}]\!]_B)^t G (A[\![\mathbf{y}]\!]_B) = ([\![\mathbf{x}]\!]_B)^t A^t G A [\![\mathbf{y}]\!]_B \\ &= ([\![\mathbf{x}]\!]_B)^t A^t A G [\![\mathbf{y}]\!]_B = ([\![\mathbf{x}]\!]_B)^t G [\![\mathbf{y}]\!]_B = \langle \mathbf{x}, \mathbf{y} \rangle , \end{aligned}$$

and φ is an isometry. Conversely, if φ is an isometry, then we have

$$([\![\mathbf{x}]\!]_B)^t G [\![\mathbf{y}]\!]_B = \langle \mathbf{x}, \mathbf{y} \rangle = \langle \varphi(\mathbf{x}), \varphi(\mathbf{y}) \rangle = ([\![\mathbf{x}]\!]_B)^t A^t G A [\![\mathbf{y}]\!]_B .$$

Since this holds for all vectors $\mathbf{x}, \mathbf{y}$, we have

$$G = A^t G A = A^{-1} G A ,$$

which implies $AG = GA$.

It remains to prove that we have $AG = GA$ if and only if $G = cE_2$. Clearly, if $G = cE_2$ then we have $AG = GA$. Conversely, assume that $AG = GA$. Since the scalar product is symmetric, so is its matrix. Hence,

$$G = \begin{pmatrix} a & b \\ b & c \end{pmatrix} ,$$

for suitable $a, b, c \in \mathbb{R}$. We obtain

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

This gives the following equations:

$$\begin{aligned} a\cos\theta - b\sin\theta &= a\cos\theta + b\sin\theta , \\ b\cos\theta - c\sin\theta &= -a\sin\theta + b\cos\theta , \\ a\sin\theta + b\cos\theta &= b\cos\theta + c\sin\theta , \\ b\sin\theta + c\cos\theta &= -b\sin\theta + c\cos\theta . \end{aligned}$$

The last equation simplifies to $2b\sin\theta = 0$. Since $\theta \neq 0, \pi$ this implies that $b = 0$. Hence, the second equation simplifies to $c\sin\theta = a\sin\theta$. Since $\sin\theta \neq 0$ it follows that $a = c$. As desired, we obtain

$$G = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} .$$