Linear Algebra II
Exercise Sheet no. 9

TECHNISCHE UNIVERSITÄT DARMSTADT

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

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Exercise 1 (Warm-up: Isomorphisms of unitary (euclidean) spaces)

(a) Let V and W be euclidean (unitary) spaces of dimension n and $\varphi \in \mathrm{Hom}(V,W)$. Show that the following are equivalent:

i. φ is an isomorphism of euclidean (unitary) spaces.

ii. $[\![\varphi]\!]^B_{B'} \in O(n)$ for some choice of orthonormal bases B of V and B' of W.

iii. $[\![\varphi]\!]^B_{B'} \in O(n)$ for every orthonormal bases B of V and B' of W.

(b) Conclude that $\varphi \in \mathrm{Hom}(V,V)$ is an orthogonal (unitary) endomorphism of the n-dimensional euclidean (unitary) space V iff $[\![\varphi]\!]^B_{B'} \in O(n)$ for some (every) combination of orthonormal bases B and B' of V.

(NB: in one direction this extends Prop. 2.3.15 in the notes.)

Solution:

a) We consider the unitary case; the euclidean case follows similarly.

Let $B = (\mathbf{v}_1,\dots,\mathbf{v}_n)$ be an orthonormal basis of V, $B' = (\mathbf{c}_1,\dots,\mathbf{c}_n)$ be an orthonormal basis of W and $A := [\![\varphi]\!]^B_{B'} = (a_{ij})$. Then for all $i,j = 1,\dots,n$,

$$\begin{aligned}\langle\varphi(\mathbf{v}_i),\varphi(\mathbf{v}_j)\rangle &= \left\langle \sum_k a_{ki}\mathbf{c}_k, \sum_l a_{lj}\mathbf{c}_l \right\rangle = \sum_{k,l} \overline{a_{ki}} a_{lj} \langle \mathbf{c}_k, \mathbf{c}_l\rangle \\ &= \sum_{k,l} \overline{a_{ki}} a_{lj}\delta_{kl} = \sum_k \overline{a_{ki}} a_{kj} = \sum_k a^+_{ik} a_{kj}\end{aligned}$$

is the entry in position i,j of A^+A.

By Lemma 2.3.9, we get that:

φ is an isomorphism of unitary spaces

$\Updownarrow$

for some (any) orthonormal basis $B = (\mathbf{v}_1,\dots,\mathbf{v}_n)$ of V, $\varphi(B) = (\varphi(\mathbf{v}_1),\dots,\varphi(\mathbf{v}_n))$ is an orthonormal basis of W

$\Updownarrow$

for some (any) orthonormal basis $B = (\mathbf{v}_1,\dots,\mathbf{v}_n)$ of V, $\langle\varphi(\mathbf{v}_i),\varphi(\mathbf{v}_j)\rangle = \delta_{ij}$ for all $i,j = 1,\dots,n$

$\Updownarrow$

for some (any) orthonormal bases $B = (\mathbf{v}_1,\dots,\mathbf{v}_n)$ of V and $B' = (\mathbf{c}_1,\dots,\mathbf{c}_n)$ of W, $\sum_k a^+_{ik}a_{kj} = \delta_{ij}$ for all $i,j = 1,\dots,n$, where $A = [\![\varphi]\!]^B_{B'}$

$\Updownarrow$

for some (any) orthonormal bases $B = (\mathbf{v}_1,\dots,\mathbf{v}_n)$ of V and $B' = (\mathbf{c}_1,\dots,\mathbf{c}_n)$ of W, $A^+A = E_n$, where $A = [\![\varphi]\!]^B_{B'}$

$\Updownarrow$

for some (any) orthonormal bases $B = (\mathbf{v}_1,\dots,\mathbf{v}_n)$ of V and $B' = (\mathbf{c}_1,\dots,\mathbf{c}_n)$ of W, $A = [\![\varphi]\!]^B_{B'} \in O(n)$.

b) Apply (a) with $W := V$.

Exercise 2 (Composition of two orthogonal projections)

(*Exercise 2.3.4 on page 68 of the notes.*) Let U and W be two subspaces of a finite dimensional euclidean or unitary vector space V, with orthogonal projections π_U and π_W onto U and W, respectively.

Prove that the following statements are equivalent:

(a) π_U and π_W commute.

(b) $\pi_W \circ \pi_U = \pi_{U \cap W}$.

(c) $\pi_W \circ \pi_U$ is an orthogonal projection.

(d) $U = (U \cap W) \oplus (U \cap W^\perp)$.

(e) $W = (U \cap W) \oplus (U^\perp \cap W)$.

Solution:
Firstly, let us remark that, using Exercise E 8.1.(e), we get that

$$\ker(\pi_W \circ \pi_U) = (\pi_W \circ \pi_U)^{-1}(\mathbf{0}) = \pi_U^{-1}\big(\pi_W^{-1}(\mathbf{0})\big) = \pi_U^{-1}(W^\perp) = (U \cap W^\perp) \oplus U^\perp.$$

(a) ⇒ (c). Since π_U and π_W commute, we have that

$$(\pi_W \circ \pi_U) \circ (\pi_W \circ \pi_U) = (\pi_W \circ \pi_W) \circ (\pi_U \circ \pi_U) = \pi_W \circ \pi_U.$$

By Exercise E 8.4.(b), it is now sufficient to show that $\ker(\pi_W \circ \pi_U) \perp \mathrm{image}(\pi_W \circ \pi_U)$.

Since $\mathrm{image}(\pi_W \circ \pi_U) \subseteq \mathrm{image}(\pi_W) \subseteq W$ and $\mathrm{image}(\pi_W \circ \pi_U) = \mathrm{image}(\pi_U \circ \pi_W) \subseteq \mathrm{image}(\pi_U) \subseteq U$, we get that

$$\mathrm{image}(\pi_W \circ \pi_U) \subseteq U \cap W.$$

Furthermore,

$$\ker(\pi_W \circ \pi_U) \quad = \quad (U \cap W^\perp) \oplus U^\perp \subseteq W^\perp + U^\perp = U^\perp + W^\perp.$$

On the other hand, $U^\perp + W^\perp \perp U \cap W$, since, by Exercise E 8.2.(d), $U^\perp + W^\perp = (U \cap W)^\perp$. It follows that $\ker(\pi_W \circ \pi_U) \perp \mathrm{image}(\pi_W \circ \pi_U)$.

(c) ⇒ (b). Assume that $\pi_W \circ \pi_U$ is an orthogonal projection. Then it is an orthogonal projection onto $\mathrm{image}(\pi_W \circ \pi_U)$. We need to prove that

$$\mathrm{image}(\pi_W \circ \pi_U) = U \cap W.$$

Since $\pi_W \circ \pi_U$ is the identity on $U \cap W$, it follows that $U \cap W \subseteq \mathrm{image}(\pi_W \circ \pi_U)$. Furthermore, $\mathrm{image}(\pi_W \circ \pi_U) \subseteq W$. Thus, it remains to show that $\mathrm{image}(\pi_W \circ \pi_U) \subseteq U$. Using the fact that $U^\perp \subseteq (U \cap W^\perp) \oplus U^\perp = \ker(\pi_W \circ \pi_U)$, we get that

$$\begin{array}{rcll} \mathrm{image}(\pi_W \circ \pi_U) & = & \big(\mathrm{image}(\pi_W \circ \pi_U)^\perp\big)^\perp & \text{by Exercise E 8.2.(c)} \\ & = & \ker(\pi_W \circ \pi_U)^\perp & \text{by Exercise E 8.1} \\ & \subseteq & (U^\perp)^\perp & \text{by Exercise E 8.2.(a), as } U^\perp \subseteq \ker(\pi_W \circ \pi_U) \\ & = & U & \text{by Exercise E 8.2.(c).} \end{array}$$

(b) ⇒ (d). Let us remark that the sum $(U \cap W) + (U \cap W^\perp)$ is direct, as $W \cap W^\perp = \{\mathbf{0}\}$.

"⊇" is obvious.

"⊇" Let $\mathbf{u} \in U$. Then $\pi_U(\mathbf{u}) = \mathbf{u}$, hence

$$\pi_W(\mathbf{u}) = (\pi_W \circ \pi_U)(\mathbf{u}) \in U \cap W.$$

Furthermore, $\mathbf{u} - \pi_W(\mathbf{u}) \in U \cap W^\perp$, since $\mathbf{u}, \pi_W(\mathbf{u}) \in U$ and $\mathbf{u} - \pi_W(\mathbf{u}) \perp W$ by Exercise E 8.4.(d). It follows that

$$\mathbf{u} = \pi_W(\mathbf{u}) + \big(\mathbf{u} - \pi_W(\mathbf{u})\big) \in (U \cap W) \oplus (U \cap W^\perp).$$

(d) ⇒ (e). Assume that $U = (U \cap W) \oplus (U \cap W^\perp)$. Then

$$V = U \oplus U^\perp = (U \cap W) \oplus (U \cap W^\perp) \oplus U^\perp.$$

It follows that every $\mathbf{w} \in W$ can be written uniquely as $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ with $\mathbf{w}_1 \in U \cap W$, $\mathbf{w}_2 \in U \cap W^\perp$ and $\mathbf{w}_3 \in U^\perp$. As $\mathbf{w}_2 \in W^\perp$, we get that

$$0 = \langle \mathbf{w}_2, \mathbf{w} \rangle = \langle \mathbf{w}_2, \mathbf{w}_1 \rangle + \langle \mathbf{w}_2, \mathbf{w}_2 \rangle + \langle \mathbf{w}_2, \mathbf{w}_3 \rangle = 0 + \langle \mathbf{w}_2, \mathbf{w}_2 \rangle + 0 = \langle \mathbf{w}_2, \mathbf{w}_2 \rangle,$$

hence $\mathbf{w}_2 = \mathbf{0}$. Since $\mathbf{w}, \mathbf{w}_1 \in W$, we have that $\mathbf{w}_3 = \mathbf{w} - \mathbf{w}_1 \in W$ too. Therefore,

$$\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_3 \in (U \cap W) \oplus (U^\perp \cap W).$$

(e) $\Rightarrow$ (d) follows by symmetry.

(d) $\Rightarrow$ (c). Again, when $U = (U \cap W) \oplus (U \cap W^{\perp})$, then

$$V = (U \cap W) \oplus (U \cap W^{\perp}) \oplus U^{\perp}.$$

We shall prove that $\pi_W \circ \pi_U$ is an orthogonal projection using Exercise E 8.4.(c). Let $\mathbf{v} \in V$ be arbitrary. Then $\mathbf{v}$ can be written uniquely as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ with $\mathbf{v}_1 \in U \cap W$, $\mathbf{v}_2 \in U \cap W^{\perp}$ and $\mathbf{v}_3 \in U^{\perp}$. It follows that

$$\begin{aligned}(\pi_W \circ \pi_U)(\mathbf{v}) &= \pi_W(\pi_U(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)) = \pi_W(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1,\\ (\pi_W \circ \pi_U)(\mathbf{v}_1) &= \pi_W(\pi_U(\mathbf{v}_1)) = \pi_W(\mathbf{v}_1) = \mathbf{v}_1,\end{aligned}$$

hence

$$\begin{aligned}((\pi_W \circ \pi_U) \circ (\pi_W \circ \pi_U))(\mathbf{v}) &= (\pi_W \circ \pi_U)((\pi_W \circ \pi_U)(\mathbf{v})) = (\pi_W \circ \pi_U)(\mathbf{v}_1)\\ &= \mathbf{v}_1 = (\pi_W \circ \pi_U)(\mathbf{v}).\end{aligned}$$

On the other hand,

$$\langle \mathbf{v} - (\pi_W \circ \pi_U)(\mathbf{v}), (\pi_W \circ \pi_U)(\mathbf{v}) \rangle = \langle \mathbf{v} - \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle + \langle \mathbf{v}_3, \mathbf{v}_1 \rangle = 0 + 0 = 0,$$

Hence, $\pi_W \circ \pi_U$ is an orthogonal projection by Exercise E 8.4.(c).

(d) $\Rightarrow$ (a). Since (d) $\Rightarrow$ (c) $\Rightarrow$ (b), it follows that (d) implies that $\pi_W \circ \pi_U = \pi_{U \cap W}$. But then (e) implies $\pi_U \circ \pi_W = \pi_{U \cap W}$, by symmetry. Since (d) is equivalent with (e), we get that (d) implies that $\pi_W \circ \pi_U = \pi_{U \cap W} = \pi_U \circ \pi_W$, and, in particular, that π_U and π_W commute.

Exercise 3 (Endomorphisms that preserve orthogonality)

Let V be a finite dimensional euclidean space. Determine all endomorphisms φ of V that preserve orthogonality, that is for which:

$$\mathbf{v} \perp \mathbf{w} \Rightarrow \varphi(\mathbf{v}) \perp \varphi(\mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in V.$$

Solution:

Let $\varphi : V \to V$ be an endomorphism that preserves orthogonality. Let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an orthonormal basis of V and $\mathbf{c}_j = \varphi(\mathbf{v}_j)$ for all $j \in \{1, \ldots, n\}$. Assume $j, k \in \{1, \ldots, n\}$ with $j \neq k$. Then $\mathbf{c}_j \perp \mathbf{c}_k$ and, since $\mathbf{v}_j + \mathbf{v}_k \perp \mathbf{v}_j - \mathbf{v}_k$, we have also that $\mathbf{c}_j + \mathbf{c}_k \perp \mathbf{c}_j - \mathbf{c}_k$ and, therefore,

$$0 = \langle \mathbf{c}_j + \mathbf{c}_k, \mathbf{c}_j - \mathbf{c}_k \rangle = \langle \mathbf{c}_j, \mathbf{c}_j \rangle - \langle \mathbf{c}_k, \mathbf{c}_k \rangle,$$

thus $\|\mathbf{c}_j\| = \|\mathbf{c}_k\|$. Hence all vectors $\mathbf{c}_j$ have the same length $s \geq 0$. If $s = 0$, then $\varphi = 0$. Otherwise, vectors $\frac{1}{s}\mathbf{c}_1, \ldots, \frac{1}{s}\mathbf{c}_n$ form an orthonormal basis of V, since $n = \dim(V)$. Lemma 2.3.9 shows that $\frac{1}{s}\varphi$ is an orthogonal map.

Thus, every endomorphism that preserves orthogonality is a scalar multiple of an orthogonal map. Conversely, it is obvious that any scalar multiple of an orthogonal map preserves orthogonality.

Exercise 4 (Jordan normal form and real matrices)

Let $A \in \mathbb{R}^{(n,n)}$ where $n = 2m$ is even. Assume that the characteristic polynomial of A is $p_A = p_0^m$, where $p_0 \in \mathbb{R}[X]$ is an irreducible polynomial of degree 2 in $\mathbb{R}[X]$ (e.g., $p_0 = X^2 + 1$). Hence p_0 splits into linear factors $(\lambda - X)(\overline{\lambda} - X)$ in $\mathbb{C}[X]$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(a) Show that if $\mathbf{v}$ is a generalised eigenvector for λ with height k, then $\overline{\mathbf{v}}$ is a generalised eigenvector for $\overline{\lambda}$ with height k, and $[\![\mathbf{v}]\!] \cap [\![\overline{\mathbf{v}}]\!] = 0$. (*Hint.* Use Lemma 1.5.6.)

(b) Show that A is similar to a real matrix $K \in \mathbb{R}^{(n,n)}$ composed of just three kinds of (2×2)-blocks: $\mathbf{0} \in \mathbb{R}^{(2,2)}$, $E_2 \in \mathbb{R}^{(2,2)}$ and some $A_0 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{R}^{(2,2)}$ with $b \neq 0$, where A_0 occurs along the diagonal, E_n and $\mathbf{0}$ immediately above the diagonal and just $\mathbf{0}$ everywhere else (a "block Jordan normal form").

Hint. Put A into Jordan normal form over $\mathbb{C}$ w.r.t. basis consisting of complex conjugate vector pairs; then combine such pairs to find a real basis.

(c) Give examples of $A_k \in \mathbb{R}^{(6,6)}$ with characteristic polynomial $(X^2 + 1)^3$ and minimal polynomials $q_{A_k} = (X^2 + 1)^k$ for $k = 1, 2, 3$.

Solution:

a) The first statement follows from the equivalence

$$(A-\lambda E)^k\mathbf{v}=\mathbf{0} \Leftrightarrow (A-\overline{\lambda}E)^k\overline{\mathbf{v}}=\mathbf{0}\,.$$

To prove that $[\![\mathbf{v}]\!] \cap [\![\overline{\mathbf{v}}]\!] = 0$, observe that if $\mathbf{w} \in [\![\mathbf{v}]\!] \cap [\![\overline{\mathbf{v}}]\!]$, then $\mathbf{w} \in \ker(\varphi - \lambda \mathrm{id})^k$ and $\mathbf{w} \in \ker(\varphi - \overline{\lambda}\mathrm{id})^k$. As the polynomials $(X-\lambda)^k$ and $(X-\overline{\lambda})^k$ are relatively prime, this can only happen for $\mathbf{w}=\mathbf{0}$, by Lemma 1.5.6.

b) By (a), we may assume that if $\mathbf{v}$ generates a Jordan block for eigenvalue λ, then $\overline{\mathbf{v}}$ generates a Jordan block of the same size for eigenvalue $\overline{\lambda}$. So, in fact, when

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

is a certain Jordan block of the matrix associated to vectors $(\mathbf{b}_1,\dots,\mathbf{b}_n)$, then also the Jordan block

$$\begin{pmatrix} \overline{\lambda} & 1 & & 0 \\ & \overline{\lambda} & \ddots & \\ & & \ddots & 1 \\ 0 & & & \overline{\lambda} \end{pmatrix}$$

appears, and we may assume it is associated to the vectors $(\overline{\mathbf{b}}_1,\dots,\overline{\mathbf{b}}_n)$ (proof: if the last vector for this basis is $\overline{\mathbf{b}}_n$, then the $(n-1)$th vector is $(A-\overline{\lambda})\overline{\mathbf{b}}_n = \overline{(A-\lambda)\mathbf{b}_n} = \overline{\mathbf{b}}_{n-1}$, and so on.).

We may rearrange these vectors as

$$(\mathbf{b}_1+\overline{\mathbf{b}}_1,\ i(\mathbf{b}_1-\overline{\mathbf{b}}_1),\ \dots,\mathbf{b}_n+\overline{\mathbf{b}}_n,\ i(\mathbf{b}_n-\overline{\mathbf{b}}_n))\,.$$

With respect to this basis of real vectors, the matrix looks as follows:

$$\left(\begin{array}{cc|cc|cc|cc} a & -b & 1 & 0 & & & & 0 \\ b & a & 0 & 1 & & & & \\ \hline & & a & -b & \ddots & \ddots & & \\ & & b & a & \ddots & \ddots & & \\ \hline & & & & \ddots & \ddots & 1 & 0 \\ & & & & \ddots & \ddots & 0 & 1 \\ \hline & & & & & & a & -b \\ 0 & & & & & & b & a \end{array}\right),$$

where $\lambda = a+ib$. From this the statement in the exercise follows.

c) We have, for example:

$$A_1=\left(\begin{array}{cc|cc|cc} 0 & -1 & & & & 0 \\ 1 & 0 & & & & \\ \hline & & 0 & -1 & & \\ & & 1 & 0 & & \\ \hline & & & & 0 & -1 \\ 0 & & & & 1 & 0 \end{array}\right), \quad A_2=\left(\begin{array}{cc|cc|cc} 0 & -1 & 1 & 0 & & 0 \\ 1 & 0 & 0 & 1 & & \\ \hline & & 0 & -1 & & \\ & & 1 & 0 & & \\ \hline & & & & 0 & -1 \\ 0 & & & & 1 & 0 \end{array}\right),$$

$$A_3=\left(\begin{array}{cc|cc|cc} 0 & -1 & 1 & 0 & & 0 \\ 1 & 0 & 0 & 1 & & \\ \hline & & 0 & -1 & 1 & 0 \\ & & 1 & 0 & 0 & 1 \\ \hline & & & & 0 & -1 \\ 0 & & & & 1 & 0 \end{array}\right).$$