Linear Algebra II
Exercise Sheet no. 8

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
May 30, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise 1 (Warm-up: orthogonal complement and orthogonal projection)

Let V be a euclidean or unitary vector space of finite dimension, U be a subspace of V and $\pi_U : V \to U$ be the orthogonal projection onto U. Check the following facts.

(a) $U^\perp$ is a subspace of V.

(b) If B is a basis of U, then $U^\perp = \{\mathbf{v} \in V \mid \mathbf{v} \perp B\}$.

(c) π_U is linear, surjective and $ker(\pi_U) = U^\perp$.

(d) $\pi_U \circ \pi_U = \pi_U$.

(e) For any subspace W of V,

$$\pi_U^{-1}(W) = (U \cap W) \oplus U^\perp.$$

(f) If $B = (\mathbf{v}_1, \ldots, , \mathbf{v}_n)$ is an orthonormal basis of U, then

$$\pi_U(\mathbf{v}) = \sum_{i=1}^{n} \langle \mathbf{v}_i, \mathbf{v} \rangle \mathbf{v}_i.$$

Solution:
In the sequel, $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$.

a) Let $\alpha, \beta \in \mathbb{F}$ and $\mathbf{v}, \mathbf{w} \in U^\perp$. For every $\mathbf{u} \in U$,

$$\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

hence $\alpha\mathbf{v} + \beta\mathbf{w} \in U^\perp$.

b) Let B be a basis of U.

"$\subseteq$" is obvious.

"$\supseteq$" Let $\mathbf{v} \perp B$. If $\mathbf{u} \in U$ is arbitrary, then $\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$ for $\lambda_i \in \mathbb{F}$, $i = 1, \ldots, n$. It follows that

$$\langle \mathbf{v}, \mathbf{u} \rangle \quad = \quad \left\langle \mathbf{v}, \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \right\rangle = \sum_{i=1}^{n} \lambda_i \langle \mathbf{v}, \mathbf{v}_i \rangle = 0,$$

hence $\mathbf{v} \in U^\perp$.

c) Let $\alpha, \beta \in \mathbb{F}$ and $\mathbf{v}, \mathbf{w} \in U^\perp$. Since, by Lemma 2.3.7, $V = U \oplus U^\perp$, there are unique $\mathbf{v}_1, \mathbf{w}_1 \in U$ and $\mathbf{v}_2, \mathbf{w}_2 \in U^\perp$ such that

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2.$$

It follows that

$$\alpha\mathbf{v} + \beta\mathbf{w} = (\alpha\mathbf{v}_1 + \beta\mathbf{w}_1) + (\alpha\mathbf{v}_2 + \beta\mathbf{w}_2) \quad \text{with } \alpha\mathbf{v}_1 + \beta\mathbf{w}_1 \in U,\ \alpha\mathbf{v}_2 + \beta\mathbf{w}_2 \in U^\perp.$$

Therefore,

$$\pi_U(\alpha \mathbf{v} + \beta \mathbf{w}) = \alpha \mathbf{v}_1 + \beta \mathbf{w}_1 = \alpha \pi_U(\mathbf{v}) + \beta \pi_U(\mathbf{w}),$$

that is, π_U is linear.

Since $\pi_U(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u} \in U$, we get that π_U is surjective.

It remains to prove that $\ker(\pi_U) = U^\perp$. This follows immediately from the fact that any $\mathbf{v} \in V$ has a unique decomposition as

$$\mathbf{v} = \pi_U(\mathbf{v}) + \mathbf{v}' \quad \text{where } \mathbf{v}' \in U^\perp.$$

d) For any $\mathbf{v} \in V$, we have that $\pi_U(\mathbf{v}) \in U$, hence $\pi_U(\pi_U(\mathbf{v})) = \pi_U(\mathbf{v})$.

e) By double inclusion. If $\mathbf{v} = \mathbf{w} + \mathbf{u}' \in (U \cap W) \oplus U^\perp$, then $\pi_U(\mathbf{v}) = \pi_U(\mathbf{w}) \in W$. Conversely, if $\pi_U(\mathbf{v}) \in W$, then $v = \pi_U(\mathbf{v}) + (\mathbf{v} - \pi_U(\mathbf{v})) \in (U \cap W) \oplus U^\perp$.

f) Let $\mathbf{u} := \sum_{i=1}^n \langle \mathbf{v}_i, \mathbf{v} \rangle \mathbf{v}_i \in U$ and $\mathbf{w} := \mathbf{v} - \mathbf{u} = \mathbf{v} - \sum_{i=1}^n \langle \mathbf{v}_i, \mathbf{v} \rangle \mathbf{v}_i$. Then

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad \text{with } \mathbf{u} \in U \text{ and } \mathbf{w} \in U^\perp.$$

It follows that $\pi_U(\mathbf{v}) = \mathbf{u}$.

Exercise 2 (Orthogonal complements)

(*Exercise 2.3.5 on page 68 of the notes.*) Let V be a euclidean or unitary vector space of finite dimension. Moreover, let U, U_1, U_2 be subspaces of V. Prove the following facts.

(a) $U_1 \subseteq U_2$ implies $U_2^\perp \subseteq U_1^\perp$.

(b) $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$.

(c) $(U^\perp)^\perp = U$.

(d) $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$.

Solution:

a)

$$\begin{aligned} \mathbf{v} \in U_2^\perp &\Rightarrow \mathbf{v} \perp \mathbf{u} \text{ for all } \mathbf{u} \in U_2 \\ &\Rightarrow \mathbf{v} \perp \mathbf{u} \text{ for all } \mathbf{u} \in U_1, \text{ as } U_1 \subseteq U_2 \\ &\Rightarrow \mathbf{v} \in U_1^\perp. \end{aligned}$$

b) "$\subseteq$" By (i), since $U_1, U_2 \subseteq U_1 + U_2$.

"$\supseteq$" Let $\mathbf{v} \in U_1^\perp \cap U_2^\perp$, and $\mathbf{u} \in U_1 + U_2$, so $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ for some $\mathbf{u}_i \in U_i$. Then

$$\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u}_1 \rangle + \langle \mathbf{v}, \mathbf{u}_2 \rangle = 0 + 0 = 0.$$

Thus $\mathbf{v} \in (U_1 + U_2)^\perp$.

c) It is easy to see that $U \subseteq (U^\perp)^\perp$. By Lemma 2.3.7, we have that $V = W^\perp \oplus W$ for all subspaces W of V. By letting $W := U^\perp$, we get that $V = (U^\perp)^\perp \oplus U^\perp$, so

$$\dim((U^\perp)^\perp) = \dim(V) - \dim(U^\perp) = \dim(U).$$

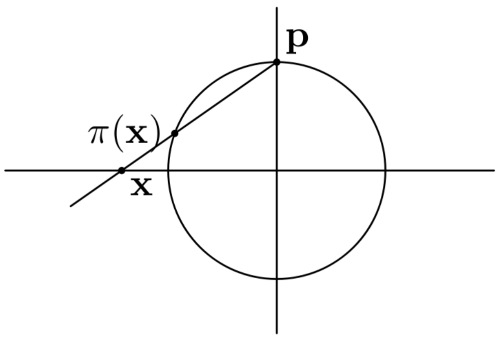
Since $U \subseteq (U^\perp)^\perp$, the equality follows.

d) Apply (ii) and (iii) to get that

$$U_1^\perp + U_2^\perp = \left(\left(U_1^\perp + U_2^\perp \right)^\perp \right)^\perp = \left(\left(U_1^\perp \right)^\perp \cap \left(U_2^\perp \right)^\perp \right)^\perp = (U_1 \cap U_2)^\perp.$$

Exercise 3 (Stereographic projection)

Let $E \subseteq \mathbb{R}^3$ be the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ and let $S \subseteq \mathbb{R}^3$ be the sphere with radius 1 and centre $\mathbf{0}$. We denote the north pole of S by $\mathbf{p} := \mathbf{e}_3$ and we set $S_* := S \setminus \{\mathbf{p}\}$.



We define a map $\pi : E \to S_*$ by letting $\pi(\mathbf{x})$ be the point of intersection between S_* and the line passing through $\mathbf{p}$ and $\mathbf{x}$.

(a) Give an explicit formula for π, i.e., find functions $f(x,y)$, $g(x,y)$, and $h(x,y)$ such that $\pi(x,y,0) = (f(x,y), g(x,y), h(x,y))$.

(b) Prove that $\pi : E \to S_*$ is a bijection.

(c) Let $C \subseteq S$ be a circle, i.e., the intersection of S with a plane given by an equation of the form $ax + by + cz = d$. Prove that the pre-image $\pi^{-1}[C]$ is either also a circle or a line.

(d) Let $c : \mathbb{R} \to E$ be a line with parametric description $x\mathbf{e}_1 + t\mathbf{v}$, $t \in \mathbb{R}$, where $\mathbf{v} = (\cos\alpha, \sin\alpha, 0)$. Note that c intersects the $\mathbf{e}_1$-axis in the point $x\mathbf{e}_1$ under the angle α. Prove that the image of c under π, i.e., the curve $\pi \circ c : \mathbb{R} \to S_*$, intersects the great circle $\{(u,0,v) \in S_* : u^2 + v^2 = 1\}$ under the same angle α. (This implies that π preserves angles. Such maps are called *conformal.*)

(*Hint.* Find the angle between the tangent vectors of the two curves. The tangent vector of a curve c at the point $c(t_0)$ is given by its derivative $\frac{\mathrm{d}}{\mathrm{d}t}c\big|_{t_0}$.)

Solution:

a) Intersecting the line with parametric description $\mathbf{p} + \lambda(\mathbf{x} - \mathbf{p})$ with S we obtain the equation

$$\|\mathbf{p} + \lambda(\mathbf{x} - \mathbf{p})\| = 1\,,$$

which has the solution

$$\lambda = \frac{2}{x^2 + y^2 + 1}\,.$$

Hence,

$$\pi(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{x^2+y^2+1}\begin{pmatrix} x \\ y \\ -1 \end{pmatrix} = \frac{1}{x^2+y^2+1}\begin{pmatrix} 2x \\ 2y \\ x^2+y^2-1 \end{pmatrix}.$$

b) To show that π is injective, suppose that $\mathbf{x}_1 = (x_1, y_1, 0)$ and $\mathbf{x}_2 = (x_2, y_2, 0)$ are two points with $\pi(\mathbf{x}_1) = \pi(\mathbf{x}_2)$. Looking at the third coordinate, we obtain the equation

$$\frac{(x_1)^2 + (y_1)^2 - 1}{(x_1)^2 + (y_1)^2 + 1} = \frac{(x_2)^2 + (y_2)^2 - 1}{(x_2)^2 + (y_2)^2 + 1}\,,$$

which implies that $(x_1)^2+(y_1)^2=(x_2)^2+(y_2)^2$. Looking at the first coordinate, we obtain

$$\frac{2x_1}{(x_1)^2+(y_1)^2+1}=\frac{2x_2}{(x_2)^2+(y_2)^2+1}\,.$$

Since the denominators are equal, it follows that $x_1=x_2$. In the same way we can show that $y_1=y_2$. Hence, $\mathbf{x}_1=\mathbf{x}_2$.

It remains to show that π is surjective. Instead of doing the calculations, let us argue geometrically. Let $\mathbf{y}\in S_*$ be an arbitrary point. We construct the line passing through $\mathbf{p}$ and $\mathbf{y}$. Since $\mathbf{y}\neq\mathbf{p}$ this line is not parallel to the plane E. Hence, it intersects E in some points $\mathbf{x}$. By definition of π it follows that $\pi(\mathbf{x})=\mathbf{y}$.

c) C is given by the two equations

$$ax+by+cz=d \quad\text{and}\quad x^2+y^2+z^2=1\,.$$

To obtain the equations for its pre-image under π we substitute the functions from (i).

$$\frac{2ax+2by+c(x^2+y^2-1)}{x^2+y^2+1}=d \quad\text{and}\quad \frac{4x^2+4y^2+(x^2+y^2-1)^2}{(x^2+y^2+1)^2}=1\,.$$

The second equation is automatically satisfied, since $\pi(\mathbf{x})\in S_*$. Hence, we only need to consider the first one. Simplifying it yields

$$2ax+2by+cx^2+cy^2-c=dx^2+dy^2+d\,,$$

which is equivalent to

$$2ax+2by+(c-d)x^2+(c-d)y^2=c+d\,.$$

If $c=d$ then we obtain

$$2ax+2by=2d\,,$$

which is the equation of a line. Otherwise, we get

$$\begin{aligned}0&=x^2+y^2+2\frac{a}{c-d}x+2\frac{b}{c-d}y-\frac{c+d}{c-d}\\&=\Big(x+\frac{a}{(c-d)}\Big)^2+\Big(y+\frac{b}{(c-d)}\Big)^2-\frac{a^2}{(c-d)^2}-\frac{b^2}{(c-d)^2}-\frac{c+d}{c-d}\,.\end{aligned}$$

This equation is of the form

$$(x-\alpha)^2+(y-\beta)^2=\gamma\,.$$

If $\gamma<0$ then it has no solution. Otherwise, it describes a circle of radius $\sqrt{\gamma}$ with centre (α,β).

d) We abbreviate $s:=\sin\alpha$ and $c:=\cos\alpha$. Note that $c^2+s^2=1$. The image is the curve

$$\begin{aligned}\pi(c(t))&=\frac{2(x+tc)}{(x+tc)^2+t^2s^2+1}\mathbf{e}_1\\&\quad+\frac{2ts}{(x+tc)^2+t^2s^2+1}\mathbf{e}_2\\&\quad+\frac{(x+tc)^2+t^2s^2-1}{(x+tc)^2+t^2s^2+1}\mathbf{e}_3\\&=\frac{2(x+tc)\mathbf{e}_1+2ts\mathbf{e}_2+((x+tc)^2+t^2s^2-1)\mathbf{e}_3}{(x+tc)^2+t^2s^2+1}\end{aligned}$$

Its derivative at $t = 0$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(c(t))\Big|_{t=0} = \frac{(2c\mathbf{e}_1 + 2s\mathbf{e}_2 + 2xc\mathbf{e}_3)(x^2+1) - (2x\mathbf{e}_1 + (x^2-1)\mathbf{e}_3)(2xc)}{(x^2+1)^2}$$
$$= \frac{2c(1-x^2)\mathbf{e}_1 + 2s(x^2+1)\mathbf{e}_2 + 4cx\mathbf{e}_3}{(x^2+1)^2}$$

The tangent vector of the great circle at the point $u\mathbf{e}_1 + v\mathbf{e}_3$ is $-v\mathbf{e}_1 + u\mathbf{e}_3$. Since

$$\pi(c(0)) = \frac{2x\mathbf{e}_1 + (x^2-1)\mathbf{e}_3}{x^2+1},$$

we compute

$$\langle \frac{-(x^2-1)\mathbf{e}_1 + 2x\mathbf{e}_3}{x^2+1}, \frac{\mathrm{d}}{\mathrm{d}t}\pi(c(t))\Big|_{t=0}\rangle = \frac{2c(1-x^2)^2 + 8cx^2}{(x^2+1)^3} = \frac{2c(x^2+1)^2}{(x^2+1)^3} = \frac{2c}{x^2+1}.$$

Furthermore, we have

$$\|\frac{\mathrm{d}}{\mathrm{d}t}\pi(c(t))\Big|_{t=0}\| = \frac{\sqrt{4c^2(1-x^2)^2 + 4s^2(x^2+1)^2 + 16c^2x^2}}{(x^2+1)^2}$$
$$= \frac{\sqrt{4c^2 - 8c^2x^2 + 4c^2x^4 + 4s^2(x^2+1)^2 + 16c^2x^2}}{(x^2+1)^2}$$
$$= \frac{\sqrt{4c^2 + 8c^2x^2 + 4c^2x^4 + 4s^2(x^2+1)^2}}{(x^2+1)^2}$$
$$= \frac{\sqrt{4c^2(x^2+1)^2 + 4s^2(x^2+1)^2}}{(x^2+1)^2}$$
$$= \frac{2(x^2+1)}{(x^2+1)^2}$$
$$= \frac{2}{x^2+1},$$
$$\|\frac{-(x^2-1)\mathbf{e}_1 + 2x\mathbf{e}_3}{x^2+1}\| = \frac{\sqrt{(x^2-1)^2 + 4x^2}}{x^2+1} = \frac{\sqrt{(x^2+1)^2}}{x^2+1} = 1.$$

Consequently, the angle β between the two curves is

$$\cos\beta = \frac{2c}{x^2+1} \cdot \frac{x^2+1}{2} = c = \cos\alpha.$$

Exercise 4 (Characterisations of orthogonal projections)

(*Exercise 2.3.2 on page 68 of the notes.*) Let φ be an endomorphism of a finite dimensional euclidean or unitary vector space V.

Show the equivalence of the following:

(a) φ is an orthogonal projection.

(b) $\varphi \circ \varphi = \varphi$ and $\ker(\varphi) \perp \operatorname{image}(\varphi)$.

(c) $\varphi \circ \varphi = \varphi$ and $\mathbf{v} - \varphi(\mathbf{v}) \perp \varphi(\mathbf{v})$ for all $\mathbf{v} \in V$.

(d) $\mathbf{v} - \varphi(\mathbf{v}) \perp \operatorname{image}(\varphi)$ for all $\mathbf{v} \in V$.

Solution:

(a) $\Rightarrow$ (b). By Exercise E 8.1.

(b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) follow immediately from the fact that $\varphi \circ \varphi = \varphi$ implies $\mathbf{v} - \varphi(\mathbf{v}) \in \ker(\varphi)$ for all $\mathbf{v} \in V$.

(c) $\Rightarrow$ (b). Let $\mathbf{a} \in \ker(\varphi)$ and $\mathbf{v} \in \operatorname{image}(\varphi)$, hence $\varphi(\mathbf{a}) = \mathbf{0}$ and $\varphi(\mathbf{v}) = \mathbf{v}$, as $\varphi \circ \varphi = \varphi$. By letting $\mathbf{w} := \mathbf{a} + \mathbf{v}$, it follows that $\varphi(\mathbf{w}) = \varphi(\mathbf{a}) + \varphi(\mathbf{v}) = \mathbf{0} + \mathbf{v} = \mathbf{v}$. Therefore,

$$\mathbf{a} = \mathbf{w} - \mathbf{v} = \mathbf{w} - \varphi(\mathbf{w}) \perp \varphi(\mathbf{w}) = \mathbf{v}.$$

(d) $\Rightarrow$ (a). Let $U := \text{image}(\varphi)$. We shall prove that $\varphi = \pi_U$, the orthogonal projection on U. If $\mathbf{v} \in V$ is arbitrary, then $\mathbf{v} = \mathbf{u} + \mathbf{u}'$ for unique $\mathbf{u} \in U$ and $\mathbf{u}' \in U^\perp$, by Lemma 2.3.7. We have to show that $\varphi(\mathbf{v}) = \mathbf{u} = \pi_U(\mathbf{v})$. Let us remark first that $\mathbf{u} - \varphi(\mathbf{v}) \in U$. On the other hand,

$$\mathbf{u} - \varphi(\mathbf{v}) = \big(\mathbf{v} - \varphi(\mathbf{v})\big) - \mathbf{u}' \in U^\perp, \quad \text{as } \mathbf{v} - \varphi(\mathbf{v}) \in U^\perp \text{ by hypothesis.}$$

Therefore, we must have $\mathbf{u} - \varphi(\mathbf{v}) = \mathbf{0}$, that is $\varphi(\mathbf{v}) = \mathbf{u}$.

Exercise 5 (More on orthogonal projections)

(*Exercise 2.3.3 on page 68 of the notes.*) Show that the orthogonal projections of an n-dimensional euclidean or unitary vector space V are precisely those endomorphisms φ of V that are represented w.r.t. a suitable orthonormal basis by a diagonal matrix with ones and zeroes on the diagonal.

Solution:

Suppose first that there exists an orthonormal basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ in which φ is represented by a diagonal matrix with ones and zeroes on the diagonal, so $\varphi \circ \varphi = \varphi$. Letting V_0 and V_1 be the eigenspaces corresponding to eigenvalues 0 and 1, respectively, it is immediate that $V = V_0 \oplus V_1$ and $V_1^\perp = V_0$. Moreover, $\varphi|_{V_1}$ is the identity map and $\varphi|_{V_0}$ is the zero map, so $V_1 = \text{image}(\varphi)$ and $V_0 = \ker(\varphi)$. Therefore $\ker(\varphi) \perp \text{image}(\varphi)$. By condition (b) of the previous exercise, we see that φ is an orthogonal projection.

Conversely, suppose that φ is an orthogonal projection. By the same exercise, we have $\varphi \circ \varphi = \varphi$ and $\ker(\varphi) \perp \text{image}(\varphi)$. Note that $V = \ker(\varphi) \oplus \text{image}(\varphi)$. This follows from the fact that $\dim(V) = \dim(\ker(\varphi)) + \dim(\text{image}(\varphi))$ and the fact that $\ker(\varphi) \perp \text{image}(\varphi)$, which implies that $\ker(\varphi) \cap \text{image}(\varphi) = \{0\}$. Let $k = \dim(\text{image}(\varphi))$. Choose an orthonormal basis $(\mathbf{b}_1, \ldots, \mathbf{b}_k)$ for $\text{image}(\varphi)$ and an orthonormal basis $(\mathbf{b}_{k+1}, \ldots, \mathbf{b}_k)$ for $\ker(\varphi)$. Since $\varphi \circ \varphi = \varphi$, it follows that $\varphi|_{\text{image}(\varphi)}$ is the identity map, and $\varphi|_{\ker(\varphi)}$ is the zero map. Hence with respect to the orthonormal basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ for V, φ is represented by the diagonal matrix whose first k entries are 1 and remaining entries are 0.