Linear Algebra II
Exercise Sheet no. 6

TECHNISCHE UNIVERSITÄT DARMSTADT

Summer term 2011
May 16, 2011

Prof. Dr. Otto
Dr. Le Roux
Dr. Linshaw

Exercise 1 (Warm-up: possible Jordan normal forms)

Let $\varphi : V \to V$ be an endomorphism of a finite dimensional $\mathbb{C}$-vector space V. Which of the following situations can occur?

(a) i. V is 6-dimensional, the minimal polynomial of φ is $(X-2)^5$, and the eigenspace of 2 has dimension 3.
ii. V is 6-dimensional, the minimal polynomial of φ is $(X-2)(X-3)^2$, and the eigenspace of 2 has dimension 3.

(b) i. φ has minimal polynomial $(X-2)^4$ and there is a vector $\mathbf{v} \in V$ with height 3.
ii. φ has minimal polynomial $(X-2)^4$ and there is a vector $\mathbf{v} \in V$ with height 6.
iii. φ has minimal polynomial $(X-2)^4$, but no vector in V has height 3.

(c) i. φ has characteristic polynomial $(X-2)^6$ and $\varphi^2 - \varphi - \mathrm{id} = \mathbf{0}$.
ii. $\varphi^2 - \varphi - 2\mathrm{id} = \mathbf{0}$ and φ has eigenvalues that are not real.

(d) i. V has a φ-invariant subspace of dimension 5, 2 is the only eigenvalue of φ, but there is no $\mathbf{v} \in V$ with $\dim(\llbracket \mathbf{v} \rrbracket) = 5$.
ii. 2 is the only eigenvalue of φ, $V = \llbracket \mathbf{v} \rrbracket \oplus \llbracket \mathbf{b} \rrbracket$ with $\dim(\llbracket \mathbf{v} \rrbracket) = 5$, but the Jordan normal form for φ contains no block of size 5.

(e) i. V can be written as the direct sum of two φ-invariant subspaces of dimension 4, but there is no Jordan block of size greater than 3 in the Jordan normal form for φ.
ii. V can be written as the direct sum of two φ-invariant subspaces of dimension 4, and in the Jordan normal form of φ there is a Jordan block of size 5.

Solution:

a) i. is impossible. As the minimal polynomial is $(X-2)^5$, there must be a Jordan block of size 5. Since the eigenspace of 2 has dimension 3, there must be 3 Jordan blocks, and that simply does not fit in a 6×6-matrix.
ii. is possible. The Jordan normal form could be

$$\begin{pmatrix} 2 & & & & & 0 \\ & 2 & & & & \\ & & 2 & & & \\ & & & 3 & 1 & \\ & & & & 3 & \\ 0 & & & & & 3 \end{pmatrix}.$$

b) i. is possible. If the Jordan normal form is

$$\begin{pmatrix} 2 & 1 & & 0 \\ & 2 & 1 & \\ & & 2 & 1 \\ 0 & & & 2 \end{pmatrix}$$

with respect to a basis $(\mathbf{b}_1, \dots, \mathbf{b}_4)$, then we have $\dim(\llbracket \mathbf{b}_3 \rrbracket) = 3$.

ii. is impossible. If $\psi = \varphi - 2\mathrm{id}$, then $\psi^4 = \mathbf{0}$. Hence, no $[\![\mathbf{v}]\!]$ can have dimension greater than 4.

iii. is impossible. There must be a Jordan block of size 4 generated by a vector $\mathbf{v}$. But then $(\varphi - 2\mathrm{id})\mathbf{v}$ must have height 3.

c) i. is impossible. The minimal polynomial must be of the form $(X-2)^k$ with $1 \leq k \leq 6$, and also divide $X^2 - X - 1$. But 2 is no root of this polynomial.

ii. is impossible. If $\varphi(\mathbf{v}) = \lambda\mathbf{v}$ with $\mathbf{v} \neq 0$, then $(\varphi^2 - \varphi - 2\mathrm{id})(\mathbf{v}) = (\lambda^2 - \lambda - 2)\mathbf{v} = \mathbf{0}$. This implies $\lambda^2 - \lambda - 2 = 0$ and, therefore, $\lambda = -1$ or $\lambda = 2$. So all possible eigenvalues are real.

d) i. is possible, for instance, take $\varphi = 2\mathrm{id}$ and $V = \mathbb{C}^5$.

ii. is impossible. The Jordan normal form of φ must consist of two blocks, one of size $\dim([\![\mathbf{b}]\!])$ and one of size $\dim([\![\mathbf{v}]\!]) = 5$.

e) i. is possible. The Jordan normal form could be

$$\begin{pmatrix} \lambda & 1 & & & & & & 0 \\ & \lambda & & & & & & \\ & & \lambda & 1 & & & & \\ & & & \lambda & & & & \\ & & & & \lambda & 1 & & \\ & & & & & \lambda & & \\ & & & & & & \lambda & 1 \\ 0 & & & & & & & \lambda \end{pmatrix}$$

with respect to a basis $(\mathbf{b}_1, \ldots, \mathbf{b}_8)$. Then $\mathrm{span}(\mathbf{b}_1, \ldots, \mathbf{b}_4)$ and $\mathrm{span}(\mathbf{b}_5, \ldots, \mathbf{b}_8)$ are φ-invariant subspaces.

ii. is impossible. If $V = V_0 \oplus V_1$ and both V_0 and V_1 are φ-invariant, the Jordan normal form for φ can be obtained by joining the normal forms for the restrictions of φ to V_0 and V_1. So if V_0 and V_1 can be chosen to have dimension 4, there can be no Jordan block of size 5 in the Jordan normal form of φ.

Exercise 2 (Commuting matrices and simultaneous diagonalization)

(a) Let M_1 and M_2 be square matrices over $\mathbb{F}$, and let q_{M_1} and q_{M_2} be the corresponding minimal polynomials. Show that the minimal polynomial of the block matrix

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

is the least common multiple of q_{M_1} and q_{M_2}. (Clearly this observation generalises to block matrices with an arbitrary number of blocks).

(b) Show that M is diagonalizable if and only if both M_1 and M_2 are diagonalizable.

(c) Let A and B be diagonalizable $n \times n$ matrices over $\mathbb{F}$ that commute with each other, i.e., $AB = BA$.

i. Show that any eigenspace of A is invariant under B.

ii. Show that A and B are *simultaneously diagonalizable*, i.e., there exists a matrix C such that $C^{-1}AC$ and $C^{-1}BC$ are both diagonal matrices.

Solution:

a) Recall that $\mathrm{lcm}(q_{M_1}, q_{M_2})$ is the polynomial q characterised by the following properties:

1. $q_{M_1} | q$ and $q_{M_2} | q$.
2. If $q_{M_1} | p$ and $q_{M_2} | p$, then $q | p$.

Let q_M be the minimal polynomial of M. Since

$$q_M(M) = \begin{pmatrix} q_M(M_1) & 0 \\ 0 & q_M(M_2) \end{pmatrix} = 0,$$

it follows that $q_M(M_1) = 0$ and $q_M(M_2) = 0$. By the definition of minimal polynomial, we have $q_{M_1} | q_M$ and $q_{M_2} | q_M$. Now suppose that $q_{M_1} | p$ and $q_{M_2} | p$ for some polynomial p. Then $p(M_1) = 0$ and $p(M_2) = 0$, so

$$p(M) = \begin{pmatrix} p(M_1) & 0 \\ 0 & p(M_2) \end{pmatrix} = 0.$$

Since $p(M) = 0$, it follows that $q_M | p$, since q_M is the minimal polynomial of M. Therefore $q_M = \mathrm{lcm}(q_{M_1}, q_{M_2})$.

b) Suppose that M is diagonalizable. Then q_M splits into linear factors with multiplicity one. The same is clearly true of q_{M_1} and q_{M_2} since $q_{M_1}|q_M$ and $q_{M_2}|q_M$. Conversely, if both M_1 and M_2 are diagonalizable, q_{M_1} and q_{M_2} split into linear factors with multiplicity one. The same clearly holds for $\mathrm{lcm}(q_{M_1}, q_{M_2}) = q_M$.

c) i. Let λ be an eigenvalue of A, and let V_λ be the corresponding eigenspace. Given $\mathbf{v} \in V_\lambda$, note that $A(B\mathbf{v}) = AB\mathbf{v} = BA\mathbf{v} = B(A\mathbf{v}) = B\lambda\mathbf{v} = \lambda B\mathbf{v}$. It follows that $B\mathbf{v} \in V_\lambda$, as desired.

ii. Let λ be an eigenvalue of A, and let V_λ be the corresponding eigenspace. Choose a basis $v_1, \dots, v_m$ for V_λ. Let U be the direct sum of the remaining eigenspaces of A, and let $v_{m+1}, \dots, v_n$ be a basis for U consisting of eigenvectors of A.

Clearly $C = (v_1, \dots, v_n)$ is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A. Let S be the matrix whose columns are the vectors $v_1, \dots, v_n$, so that $A' = S^{-1}AS$ is the diagonal matrix $\mathrm{diag}(\lambda, \dots, \lambda, \mu_{m+1} \dots \mu_n)$. (Here the first m diagonal entries are λ).

Since the eigenspaces of A are invariant under B, it follows that $B' = S^{-1}BS$ is block diagonal of the form $\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$, where B_1 is an $m \times m$ block, and B_2 is an $(n-m) \times (n-m)$ block. Clearly B_1 commutes with the $m \times m$ matrix $\mathrm{diag}(\lambda, \dots, \lambda) = \lambda E_m$. Since A and B commute, it follows that A' and B' commute, which implies that B_2 commutes with the $(n-m) \times (n-m)$ matrix $\mathrm{diag}(\mu_{m+1}, \dots, \mu_n)$. By Part (b), B_1 is diagonalizable, so there exists an $m \times m$ matrix T such that $T^{-1}B_1T$ is a diagonal matrix D_1. Let U be the $n \times n$ matrix $\begin{pmatrix} T & 0 \\ 0 & E_{n-m} \end{pmatrix}$. Then $A'' = (SU)^{-1}ASU = U^{-1}A'U$ is diagonal, and

$$B'' = (SU)^{-1}BSU = U^{-1}B'U = \begin{pmatrix} D_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

We now proceed in the same way with the smaller matrix B_2.

Exercise 3 (Computing the Jordan normal form)
Let

$$A := \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & -1 & -3 & -2 \\ -2 & 3 & 5 & 2 \\ -1 & 2 & 2 & 3 \end{pmatrix}.$$

Find a regular matrix S and a matrix J in Jordan normal form such that $A = SJS^{-1}$.
Hint. The characteristic polynomial of A is $p_A = (2-X)^4$.

Solution:
2 is the only eigenvalue. Thus, we consider

$$C := A - 2E_4 = \begin{pmatrix} -1 & 2 & 2 & 1 \\ 2 & -3 & -3 & -2 \\ -2 & 3 & 3 & 2 \\ -1 & 2 & 2 & 1 \end{pmatrix}.$$

We see that $\dim(\ker(C)) = 2$, i.e., A has two linearly independent eigenvectors. This means that J consists of two Jordan blocks, either both of size 2, or one of size 3 and one of size 1. To see which of the two cases occurs, we compute C^2. As $C^2 = \mathbf{0}$, J will consist of two blocks of size two, i.e.,

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

To find the basis transformation S, we first need to determine a suitable basis. For that, we need to find two linearly independent vectors $\mathbf{u}_2$ and $\mathbf{u}_4$, such that

$$\dim([\![\mathbf{u}_2]\!]) = \dim([\![\mathbf{u}_4]\!]) = 2 \quad \text{and} \quad [\![\mathbf{u}_2]\!] \cap [\![\mathbf{u}_4]\!] = 0.$$

For example, we can take

$$\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} .$$

The other basis vectors will be

$$\mathbf{u}_1 = C\mathbf{u}_2 = \begin{pmatrix} -1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_3 = C\mathbf{u}_4 = \begin{pmatrix} 2 \\ -3 \\ 3 \\ 2 \end{pmatrix} .$$

The basis transformation S has as its columns the representations of the new basis vectors in terms of the old (or standard) basis. So the desired matrix is

$$S = \begin{pmatrix} -1 & 1 & 2 & 0 \\ 2 & 0 & -3 & 1 \\ -2 & 0 & 3 & 0 \\ -1 & 0 & 2 & 0 \end{pmatrix} .$$

Exercise 4 (Exponential function for matrices)
Let

$$J_\lambda := \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \in \mathbb{C}^{n \times n}$$

be a Jordan block with eigenvalue λ. For an arbitrary matrix A, we define

$$e^A := \sum_{i=0}^{\infty} \frac{A^i}{i!} .$$

(a) Compute J_0^k.

(b) Compute J_λ^k. *Hint.* Use the decomposition $J_\lambda = \lambda E_n + J_0$.

For the following we leave aside all the convergence issues. It is indeed safe here, but not part of linear algebra.

(c) Suppose that A and B are matrices with $AB = BA$. Show that $e^{A+B} = e^A e^B$.

(d) Show that $e^{S^{-1}AS} = S^{-1} e^A S$, for an arbitrary matrix A and an invertible one S.

(e) Prove that

$$e^{J_\lambda} = e^\lambda \sum_{i=0}^{n-1} \frac{J_0^i}{i!} .$$

Solution:

a)

$$J_0^k = \begin{pmatrix} \overbrace{0 & \cdots & 0}^{k} & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

b)

$$J_\lambda^k = (\lambda E_n + J_0)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^i J_0^{k-i}.$$

c)

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \binom{k}{i} \frac{A^i B^{k-i}}{k!} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{A^i B^{k-i}}{i!\,(k-i)!} = \sum_{k=0}^{\infty} \frac{A^k}{k!} \cdot \sum_{i=0}^{\infty} \frac{B^i}{i!} = e^A e^B.$$

d) We have $(S^{-1}AS)^k = S^{-1}ASS^{-1}AS \cdots S^{-1}AS = S^{-1}AA \cdots AS = S^{-1}A^k S$. Consequently,

$$e^{S^{-1}AS} = \sum_{i=0}^{\infty} \frac{(S^{-1}AS)^i}{i!} = S^{-1}\Big[\sum_{i=0}^{\infty} \frac{A^i}{i!}\Big] S = S^{-1} e^A S\,.$$

e)

$$e^{J_\lambda} = e^{\lambda E_n + J_0} = e^{\lambda E_n} e^{J_0} = e^{\lambda} \sum_{i=0}^{\infty} \frac{J_0^i}{i!} = e^{\lambda} \sum_{i=0}^{n-1} \frac{J_0^i}{i!}\,.$$

Exercise 5 (Square roots)

(a) Let $a_0, \ldots, a_{n-1} \in \mathbb{C}$ and let N be the $n \times n$ matrix $\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & \dots & 0 \end{pmatrix}$. When is $(\sum_{i=0}^{n-1} a_i N^i)^2$ a Jordan block?

(b) Deduce a sufficient condition for $dE_n + N \in \mathbb{C}^{(n,n)}$ to have a square root.

(c) Deduce a sufficient condition for complex matrices to have complex square roots.

Remark: using techniques from Lie group theory, which combine differential geometry, topology and group theory, one can also obtain that the exponential map on matrices, $A \mapsto e^A$, is a surjection of $\mathbb{C}^{(n,n)}$ onto $GL_n(\mathbb{C})$. It follows that the equality $[e^{\frac{1}{2}A}]^2 = e^A$ yields square roots for any regular matrix.

Solution:

a) First note that generally speaking $(\sum_{i=0}^{k} a_i)^2 = \sum_{0 \le i,j \le k} a_i a_j$ whenever $a_0, \ldots, a_k$ are elements of a ring.

$$\begin{aligned} A := (\sum_{i=0}^{n-1} a_i N^i)^2 &= \sum_{0 \le i,j \le n-1} a_i a_j N^{i+j} \\ &= \sum_{0 \le k \le n-1} d_k N^k \qquad \text{where } d_k := \sum_{i=0}^{k} a_i a_{k-i} \end{aligned}$$

For A to be a Jordan block, d_1 must equal 1, that is, $2a_0 a_1 = 1$. Therefore $a_0 \neq 0$ and $a_1 = \frac{1}{2a_0}$ are necessary. Moreover d_k must be zero for $1 < k$, that is, $a_k = \frac{-\sum_{i=1}^{k-1} a_i a_{k-i}}{2a_0}$. These conditions are also sufficient.

b) If $d \neq 0$ then let $a_0^2 := d$ (so that $d_0 = d$), let $a_1 := \frac{1}{2a_0}$, and for all $1 < k \le n-1$ let us define the a_k by induction. $a_k := \frac{-\sum_{i=1}^{k-1} a_i a_{k-i}}{2a_0}$. We have $(\sum_{i=0}^{n-1} a_i N^i)^2 = dE_n + N$.

c) If a matrix is invertible, it has a square root. Indeed it suffices to shows that one of its JNF has a square root, by an argument similar to the previous exercise. Since the matrix is invertible, 0 is not a root of its characteristic polynomial, so all JNF blocks are of the form $dE + N$ with $d \neq 0$. Each of those has a square root by the previous question, so has the whole JNF, by block multiplication.