Linear Algebra II
Exercise Sheet no. 5

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Summer term 2011
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May 6, 2011
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Exercise 1

(a) Consider 2×2-matrices over the complex numbers. Why does their minimal polynomial determine their characteristic polynomial? Is the same true for 3×3-matrices?

(b) Find two 2×2-matrices that are not similar, but have the same characteristic polynomial.

(c) Show that any two 2×2-matrices with the same minimal polynomial are similar in $\mathbb{C}^{(2,2)}$. Is the same true in $\mathbb{R}^{(2,2)}$?

(d) Discuss necessary and sufficient conditions (also in terms of the determinant, the trace, and the minimal and characteristic polynomial of a matrix) for the similarity of two matrices. Use these criteria to split the following 9 matrices into equivalence classes w.r.t. similarity.

$$A_1 = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 3 & 2 \\ 6 & 8 & 7 \end{pmatrix} \quad A_2 = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 7 & 2 \\ 2 & 8 & 6 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad A_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad A_6 = \begin{pmatrix} 2 & 4 & 3 \\ 3 & 1 & 2 \\ 8 & 6 & 7 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 4 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix} \quad A_8 = \begin{pmatrix} 2 & 5 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_9 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution:

a) If the minimal polynomial has degree 2, then it must be the characteristic polynomial. If it has degree 1, it must consist of one linear factor $(X - \lambda)$, and the characteristic polynomial is $(X - \lambda)^2$. (And if it is the polynomial $p = 0$, then so is the characteristic polynomial.)

A 3×3-matrix with minimal polynomial $(X-2)(X-3)$ may have characteristic polynomial $(X-2)^2(X-3)$ or $(X-2)(X-3)^2$. (Try to find an example of both!)

b) The following matrices have the same characteristic, but different minimal polynomials, and are therefore not similar:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

c) To prove that matrices in $\mathbb{C}^{(2,2)}$ with the same minimal polynomial are similar, we find for each polynomial p of degree two a matrix to which all matrices with p as minimal polynomial are similar. So, if the minimal polynomial of a matrix is of the form $(X - \lambda)(X - \mu)$ with $\lambda \neq \mu$, then this also the characteristic polynomial, and the matrix is similar to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

If the minimal polynomial is $(X-\lambda)$, then the matrix is (similar to)

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

and if the minimal polynomial is $(X-\lambda)^2$, then it is similar to

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

(This will later follow from the Jordan Normal Form Theorem).

In $\mathbb{R}^{(2,2)}$ we have the additional possibility of a minimal polynomial of degree two that is irreducible over $\mathbb{R}$. Over $\mathbb{C}$ the polynomial splits as $(X-\lambda)(X-\overline{\lambda})$ with $\lambda \neq \overline{\lambda}$. Writing $\lambda = re^{i\varphi}$ it follows from the previous exercise that the matrix is similar in $\mathbb{R}^{(2,2)}$ to

$$r\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

So we deduce that also in $\mathbb{R}^{(2,2)}$ matrices with the same minimal polynomial are similar.

d) Similar matrices share the following features:

- They represent the same linear map with respect to different bases.
- They have the same eigenvalues with the same algebraic and geometric multiplicities.
- They have the same characteristic and minimal polynomial.
- They have the same determinant and trace.
- They have the same rank.
- One is invertible, diagonalisable, idempotent, nilpotent etc. iff the other is.

So these conditions are all necessary. We note that, if the matrices are diagonalisable, then having the same characteristic polynomial is also sufficient.

After comparing the traces and determinants it suffices to investigate the following classes for similarity:

$$\{A_1\}, \quad \{A_3\}, \quad \{A_6\}, \quad \{A_8, A_9\}, \quad \{A_2, A_4, A_5, A_7\}.$$

Since $p_{A_8} = (X-1)(X-2)(X-3) = p_{A_9}$ the matrix A_8 (and A_9) is diagonalisable. It follows that the matrices A_8 and A_9 are similar.

The characteristic polynomial of A_7 is

$$(2-X)[(-X)(4-X)+4] = (2-X)(X^2-4X+4) = (2-X)^3.$$

The eigenspace corresponding to the eigenvalue 2 has dimension 1 for A_2, dimension 2 for A_4 and A_7, and dimension 3 for A_5. The matrices A_4 and A_7 are similar, since we have $A_7 = SA_4S^{-1}$ with

$$S = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \; S^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Exercise 2 (Endomorphisms and bases)

Let $\varphi \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$ be an endomorphism of $\mathbb{R}^3$ that, for some $\lambda \in \mathbb{R}$, is represented by the matrix

$$A_\lambda := \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

(a) Check that the third basis vector in a basis B giving rise to A_λ as $A_\lambda = [\![\varphi]\!]_B^B$ must be in $\ker(\varphi - \lambda\mathrm{id})^3 \setminus \ker(\varphi - \lambda\mathrm{id})^2$.

(b) Describe in words which properties of φ guarantee that $[\![\varphi]\!]_B^B = A_\lambda$ for some basis B (for instance, in terms of eigenvalues, eigenvectors, the minimal polynomial, or the characteristic polynomial).

(c) For fixed φ (and λ), describe the set of all bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ for which $[\![\varphi]\!]_B^B = A_\lambda$.
Hint. Use φ to express $\mathbf{b}_1$ in terms of $\mathbf{b}_2$ and $\mathbf{b}_2$ in terms of $\mathbf{b}_3$, and determine the possible choices for $\mathbf{b}_3$.

(d) For $\lambda = 0$, what does the condition that $[\![\varphi]\!]_B^B = A_0$, for some basis B, tell us about dimensions of and the relationship between $\mathrm{Im}(\varphi)$ and $\ker(\varphi)$? What are the invariant subspaces?

Solution:

a) Let $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be such a basis. $p_\varphi(X) = (\lambda - X)^3$ so $(\varphi - \lambda \mathrm{id})^3 = 0$. Also, some calculation shows that $(\varphi - \lambda \mathrm{id})^2(\mathbf{b}_3) = \varphi^2(\mathbf{b}_3) - 2\lambda\varphi(\mathbf{b}_3) + \lambda^2\mathbf{b}_3 = \cdots = \mathbf{b}_1 \neq \mathbf{0}$.

b) We can find a basis B with $[\![\varphi]\!]_B^B = A_\lambda$ if, and only if, φ has only one eigenvalue λ, of geometric multiplicity 1, and its minimal polynomial is $q_\varphi = (X - \lambda)^3$.

c) If B is a basis as above, then

$$\mathbf{b}_1 = \varphi(\mathbf{b}_2) - \lambda\mathbf{b}_2 = (\varphi - \lambda \mathrm{id})\mathbf{b}_2 \quad \text{and} \quad \mathbf{b}_2 = \varphi(\mathbf{b}_3) - \lambda\mathbf{b}_3 = (\varphi - \lambda \mathrm{id})\mathbf{b}_3 .$$

Hence, every choice of $\mathbf{b}_3$ uniquely determines $\mathbf{b}_1$ and $\mathbf{b}_2$.

For which $\mathbf{b}_3$ do we get a basis this way? Clearly, we must have $\mathbf{b}_1 \neq \mathbf{0}$ and $\mathbf{b}_2 \neq \mathbf{0}$. Hence,

$$\mathbf{b}_2 \notin \ker(\varphi - \lambda \mathrm{id}) \quad \text{and} \quad \mathbf{b}_3 \notin \ker(\varphi - \lambda \mathrm{id}) .$$

We can simplify these conditions to the single necessary condition

$$\mathbf{b}_3 \notin \ker(\varphi - \lambda \mathrm{id})^2 .$$

We claim that this condition is also sufficient, i.e., for every $\mathbf{b}_3 \in \mathbb{R}^3 \setminus \ker(\varphi - \lambda \mathrm{id})^2$, the vectors $(\varphi - \lambda \mathrm{id})^2\mathbf{b}_3$, $(\varphi - \lambda \mathrm{id})\mathbf{b}_3$, $\mathbf{b}_3$ form a basis.

First, we show that $\mathbf{b}_1 := (\varphi - \lambda \mathrm{id})^2\mathbf{b}_3$ and $\mathbf{b}_2 := (\varphi - \lambda \mathrm{id})\mathbf{b}_3$ are linearly independent. For a contradiction, suppose otherwise. Then, since both vectors are non-zero, there is a non-zero scalar α such that

$$(\varphi - \lambda \mathrm{id})^2\mathbf{b}_3 = \alpha(\varphi - \lambda \mathrm{id})\mathbf{b}_3 .$$

We can rewrite this equation to

$$\varphi(\mathbf{b}_2) = (\alpha + \lambda)\mathbf{b}_2 .$$

Hence, $\mathbf{b}_2$ is an eigenvector of φ with eigenvalue $\alpha + \lambda$. Since λ is the only eigenvalue of φ, we obtain $\alpha = 0$. A contradiction.

Finally, we show that all three vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent. Otherwise, we would have $\mathbf{b}_3 \in \mathrm{span}(\mathbf{b}_1, \mathbf{b}_2)$. Since $\mathbf{b}_1, \mathbf{b}_2 \in \ker(\varphi - \lambda \mathrm{id})^2$ and $\ker(\varphi - \lambda \mathrm{id})^2$ is a subspace, it follows that

$$\mathbf{b}_3 \in \mathrm{span}(\mathbf{b}_1, \mathbf{b}_2) \subseteq \ker(\varphi - \lambda \mathrm{id})^2 .$$

Again a contradiction.

d) Let $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be a basis such that $[\![\varphi]\!] = A_0$. Then

$$\ker(\varphi) = \mathrm{span}(\mathbf{b}_1) \quad \text{and} \quad \mathrm{Im}(\varphi) = \mathrm{span}(\mathbf{b}_1, \mathbf{b}_2) .$$

Hence,

$$\ker(\varphi) \subseteq \mathrm{Im}(\varphi) , \quad \dim(\ker(\varphi)) = 1 , \quad \dim(\mathrm{Im}(\varphi)) = 2 .$$

We claim that the only invariant subspaces are $\{\mathbf{0}\}$, $\ker(\varphi)$, $\mathrm{Im}(\varphi)$, and $\mathbb{R}^3$. Let U be an invariant subspace. If U contains some vector $\mathbf{v} \notin \ker(\varphi^2)$, then $\varphi(\mathbf{v})$ and $\varphi^2(\mathbf{v})$ are also in U and $\mathbf{v}, \varphi(\mathbf{v}), \varphi^2(\mathbf{v})$ are linearly independent. Hence, $\dim(U) \geq 3$ and $U = \mathbb{R}^3$.

Similarly, if $U \subseteq \ker(\varphi^2)$ but there is some $\mathbf{v} \in U \setminus \ker(\varphi)$, then $\varphi(\mathbf{v})$ and $\mathbf{v}$ are linearly independent vectors in U. Hence, $\dim(U) = 2$ and $U = \ker(\varphi^2) = \mathrm{Im}(\varphi)$.

Finally, if $U \subseteq \ker(\varphi)$ then U is either 1-dimensional and, hence, $U = \ker(\varphi)$, or U is 0-dimensional and $U = \{\mathbf{0}\}$.

Exercise 3 (Nilpotent endomorphisms)

Recall that an endomorphism $\varphi : V \to V$ is *nilpotent* if there is some $k \in \mathrm{N}$ such that $\varphi^k = 0$. The minimal such k is called the *index* of φ.

(a) Suppose that V is $\mathrm{Pol}_n(\mathbb{R})$ the $\mathbb{R}$-vector space of all polynomial functions of degree up to n. Show that the usual differential operator $\partial : V \to V : f \mapsto f'$ is nilpotent of index $n+1$.

Suppose that $\varphi : V \to V$ is nilpotent with index k.

(b) Show that $q_\varphi = X^k$.

(c) Show that, for every $\mathbf{v} \in V$, $W := \mathrm{span}(\mathbf{v}, \varphi(\mathbf{v}), \ldots, \varphi^{k-1}(\mathbf{v}))$ is an invariant subspace.

(d) Let W be the subspace from (iii) where we additionally assume that $\varphi^{k-1}(\mathbf{v}) \neq \mathbf{0}$. Show that the restriction φ_0 of φ to W is nilpotent with index k.

(e) Suppose that V has dimension k. Show that there is some basis B such that

$$[\![\varphi]\!]_B^B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

Solution:

a) Since $\partial^{n+1}(x \mapsto x^i) = 0$, for all $i \leq n$, the index of ∂ is at most $n+1$. On the other hand, $\partial^i(x \mapsto x^n) \neq 0$, for $i \leq n$. Therefore, the index is exactly $n+1$.

b) The minimal polynomial q_φ must divide X^k since $\varphi^k = 0$. By minimality of k, we have $\varphi^i \neq 0$, for $i < k$. Hence, $q_\varphi \neq X^i$, for $i < k$. Therefore, $q_\varphi = X^k$.

c) Let $\mathbf{w} \in \mathrm{span}(\mathbf{v}, \varphi(\mathbf{v}), \ldots, \varphi^{k-1}(\mathbf{v}))$. Then

$$\mathbf{w} = \alpha_0 \mathbf{v} + \alpha_1 \varphi(\mathbf{v}) + \cdots + \alpha_{k-2} \varphi^{k-2}(\mathbf{v}) + \alpha_{k-1} \varphi^{k-1}(\mathbf{v}),$$

and

$$\begin{aligned} \varphi(\mathbf{w}) &= \alpha_0 \varphi(\mathbf{v}) + \alpha_1 \varphi^2(\mathbf{v}) + \cdots + \alpha_{k-2} \varphi^{k-1}(\mathbf{v}) + \alpha_{k-1} \varphi^k(\mathbf{v}) \\ &= \alpha_0 \varphi(\mathbf{v}) + \alpha_1 \varphi^2(\mathbf{v}) + \cdots + \alpha_{k-2} \varphi^{k-1}(\mathbf{v}) \in \mathrm{span}(\mathbf{v}, \varphi(\mathbf{v}), \ldots, \varphi^{k-1}(\mathbf{v})). \end{aligned}$$

Hence $\varphi(\mathbf{w}) \in W$, for every $\mathbf{w} \in W$.

d) For every $\mathbf{w} \in V$, we have $\varphi^k(\mathbf{w}) = 0$. This implies that $\varphi_0^k(\mathbf{w}) = 0$, for all $\mathbf{w} \in W \subseteq V$. Hence, the index is at most k. It cannot be less than k since $\mathbf{v} \in W$ and $\varphi_0^{k-1}(\mathbf{v}) = \varphi^{k-1}(\mathbf{v}) \neq \mathbf{0}$.

e) Since the index of φ is k, there is some $\mathbf{v} \in V$ such that $\varphi^{k-1}(\mathbf{v}) \neq \mathbf{0}$. For the basis $B = (\varphi^{k-1}(\mathbf{v}), \ldots, \varphi(\mathbf{v}), \mathbf{v})$, the matrix $[\![\varphi]\!]_B^B$ has the desired form.

Exercise 4 (Characteristic and minimal polynomial)

Let $A \in \mathbb{F}^{(n,n)}$ have the characteristic polynomial p_A and the minimal polynomial $q_A = X^r + \sum_{i=0}^{r-1} c_i X^i$.

(a) Let $B_0, B_1, \ldots, B_r$ be defined as below.

$$\begin{aligned} B_0 &:= E_n \\ B_1 &:= A + c_{r-1} E_n \\ B_2 &:= A^2 + c_{r-1} A + c_{r-2} E_n \\ \cdots & \\ B_{r-1} &:= A^{r-1} + c_{r-1} A^{r-2} + \cdots + c_1 E_n \\ B_r &:= A^r + c_{r-1} A^{r-1} + \cdots + c_0 E_n \end{aligned}$$

Let $B(X) := X^{r-1} B_0 + X^{r-2} B_1 + \cdots + X B_{r-2} + B_{r-1}$ and show that $(X E_n - A) B(X) = q_A(X E_n)$.

(b) Use part (a) to show that p_A divides $(q_A)^n$.

(c) Use part (b) to show that p_A and q_A have the same irreducible factors.

Solution:

a) One sees that for the sequence B_i we have:

$$\begin{aligned} B_0 &= E_n \\ B_1 - AB_0 &= A + c_{r-1}E_n - A \\ &= c_{r-1}E_n \\ &\cdots \\ B_{r-1} - AB_{r-2} &= c_1 E_n \\ B_r - AB_{r-1} &= c_0 E_n \end{aligned}$$

As $B_r = q_A(A) = 0$ we get

$$-AB_{r-1} = c_0 E_n - B_r = c_0 E_n.$$

Using these observations we can determine $(XE_n - A)B(X)$:

$$\begin{aligned} (XE_n - A)B(X) &= (X^r B_0 + X^{r-1}B_1 + \cdots + X^2 B_{r-2} + XB_{r-1}) \\ &\quad -(X^{r-1}AB_0 + X^{r-2}AB_1 + \cdots + XAB_{r_2} + AB_{r-1}) \\ &= X^r B_0 + X^{r-1}(B_1 - AB_0) + \cdots + XB_{r-1} - AB_{r-2} - AB_{r-1} \\ &= X^r E_n + X^{r-1}c_1 E_n + X^{r-2}c_{r-2}E_n + \cdots + Xc_1 E_n + c_0 E_n \\ &= q_A(XE_n) \end{aligned}$$

b) The determinant on both sides of the above equation gives $|(XE_n - A)||B(X)| = |q_A(XE_n)| = (q_A(X))^n$. Since $|B(X)|$ is a polynomial, $|XE_n - A|$ divides $(q_A)^n$; that is the characteristic polynomial p_A of A divides $(q_A)^n$.

c) Suppose f is an irreducible polynomial. If f divides q_A then, since q_A divides p_A, f divides p_A. On the other hand, if f divides q_A, then by part (a), f divides $(q_A)^n$. But f is irreducible; hence f divides q_A. Thus q_A and p_A have the same irreducible factors.