Linear Algebra II
Exercise Sheet no. 4

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Exercise 1 (Warm-up)

Prove the Cayley-Hamilton Theorem for 2×2 matrices by direct computation.

Solution:
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Clearly the characteristic polynomial $p_A(\lambda)$ is given by

$$p_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc).$$

By substituting A into $p_A(\lambda)$ we get

$$\begin{aligned} & A^2 - (a+d)A + (ad-bc)E_2 \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^2+cb & ab+bd \\ ac+dc & cb+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ca+cd & da+d^2 \end{pmatrix} + \begin{pmatrix} ad-cb & 0 \\ 0 & ad-cb \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Exercise 2 (Eigenvalues)

Let p be a polynomial in $\mathbb{F}[X]$ and $A \in \mathbb{F}^{(n,n)}$. Show that if λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue of the matrix $p(A)$.

Solution:
If $p = \sum_{k=0}^{m} a_k X^k$, then

$$p(A) = \sum_{k=0}^{m} a_k A^k = a_0 E_n + a_1 A + \ldots + a_m A^m.$$

Let $\mathbf{v} \in \mathbb{F}^n$ be an eigenvector of A with eigenvalue λ. It is easy to see by induction on k that $\mathbf{v}$ is an eigenvalue of A^k with eigenvalue λ^k for all $k \in \mathbb{N}$.

It follows that

$$\begin{aligned} p(A)\cdot\mathbf{v} &= \left(\sum_{k=0}^{m} a_k A^k\right)\mathbf{v} = \sum_{k=0}^{m} a_k (A^k\mathbf{v}) = \sum_{k=0}^{m} a_k \lambda^k \mathbf{v} = \left(\sum_{k=0}^{m} a_k \lambda^k\right)\mathbf{v} \\ &= p(\lambda)\mathbf{v} \end{aligned}$$

Exercise 3 (Trace)

Recall that $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$ is the *trace* of an $n \times n$-matrix $A = (a_{ij}) \in \mathbb{F}^{(n,n)}$.

(a) Show that for any matrices $A, B \in \mathbb{F}^{(n,n)}$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Use this to show that similar matrices have the same trace.

(b) How does the characteristic polynomial p_A of a matrix A determine $\operatorname{tr}(A)$ and $\det(A)$? From this, conclude (again) that the trace is invariant under similarity (as is the determinant, of course).

(c) Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Show that $\mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$. Therefore the trace is not invariant under arbitrary permutations of products of matrices.

Solution:

a) Since the ij-th entry $(AB)_{(ij)}$ is given by $\sum_{k=1}^n a_{ik} b_{kj}$, we have $\mathrm{tr}(AB) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki}$. Similarly, $\mathrm{tr}(BA) = \sum_{i=1}^n \sum_{k=1}^n b_{ik} a_{ki}$, so the claim is immediate.

Suppose next that $A = CBC^{-1}$ for some invertible matrix C. Since $\mathrm{tr}(CR) = \mathrm{tr}(RC)$ for $R = BC^{-1}$, we have

$$\mathrm{tr}(A) = \mathrm{tr}(CR) = \mathrm{tr}(RC) = \mathrm{tr}(BC^{-1}C) = \mathrm{tr}(B).$$

b) Let $p_A = \sum_{i=0}^n a_n X^n$ be the characteristic polynomial of A. We have

$$a_n = (-1)^n, \quad a_{n-1} = (-1)^{n-1}\mathrm{tr}(A), \quad \text{and} \quad a_0 = \det(A).$$

By Observation 1.1.7 in the lecture notes, similar matrices have the same characteristic polynomial. Hence, they also have the same trace and determinant.

c) A calculation shows that $\mathrm{tr}(ABC) = 15$ and $\mathrm{tr}(ACB) = 7$.

Exercise 4 (Characteristic polynomial)

(a) Determine the characteristic polynomial p_A of the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & a_0 \\ 1 & 0 & 0 & \dots & 0 & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & a_2 \\ 0 & 0 & 1 & \dots & 0 & 0 & a_3 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1} \end{pmatrix}, \quad \text{with } n \geq 1.$$

Hint: expand the determinant along the last column.

(b) Show that every polynomial $p \in \mathbb{F}[X]$ of degree $n \geq 1$ with leading coefficient $(-1)^n$ occurs as a characteristic polynomial of a matrix $A \in \mathbb{F}^{(n,n)}$.

Solution:

a) $p_A = (-1)^n x^n + (-1)^{n+1} \sum_{i=0}^{n-1} a_i x^i$.

b) Let $n \geq 1$ and $p = (-1)^n x^n + \sum_{i=0}^{n-1} a_i x^i$. Take

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & (-1)^{n-1}a_0 \\ 1 & 0 & 0 & \dots & 0 & 0 & (-1)^{n-1}a_1 \\ 0 & 1 & 0 & \dots & 0 & 0 & (-1)^{n-1}a_2 \\ 0 & 0 & 1 & \dots & 0 & 0 & (-1)^{n-1}a_3 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & (-1)^{n-1}a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & (-1)^{n-1}a_{n-1} \end{pmatrix}$$

By (b), we get that

$$p_A = (-1)^n x^n + (-1)^{n+1} \sum_{i=0}^{n-1} (-1)^{n-1} a_i x^i = (-1)^n x^n + \sum_{i=0}^{n-1} a_i x^i = p.$$

Exercise 5 (Polynomials over $\mathbb{F}_2$)

(a) Show that in $\mathbb{F}_2[X]$ any non-linear polynomial with an odd number of powers X^i for $i \geq 1$ (with or without the constant term 1) is reducible.

(b) Find in $\mathbb{F}_2[X]$ all irreducible polynomials of degree 3 and 4.

Solution:

a) A non-linear polynomial without the constant term 1 has the root 0, a non-linear polynomial with an odd number of powers X^i for $i \geq 1$ and the constant term 1 has the root 1. So reducibility follows from Proposition 1.2.12 on page 21 of the notes.

b) We first consider polynomials of degree 3. According to (a), the only polynomials that need to be considered are $p_1 = X^3 + X + 1$ and $p_2 = X^3 + X^2 + 1$. Both of these are irreducible, for if they were reducible, they would be divisible by a linear factor X or $X + 1$. But that is impossible, since neither 0 nor 1 is a root of either of the two polynomials.

Now consider polynomials of degree 4. According to (a), the only polynomials that come into consideration are:

$$\begin{aligned} p_1 &= X^4 + X + 1, \\ p_2 &= X^4 + X^2 + 1, \\ p_3 &= X^4 + X^3 + 1, \\ p_4 &= X^4 + X^3 + X^2 + X + 1. \end{aligned}$$

Neither of these is divisible by a linear factor, since they have no roots. So if any of these were reducible, they would factor as a product of two irreducible polynomials of degree 2. But the only irreducible polynomial of degree 2 in $\mathbb{F}_2[X]$ is $X^2 + X + 1$. So if any of the above would be reducible, it would be $(X^2 + X + 1)^2 = X^4 + X^2 + 1 = p_2$. So p_1, p_3 and p_4 are the irreducible polynomials of degree 4 in $\mathbb{F}_2[X]$.