Linear Algebra II
Exercise Sheet no. 2

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Exercise 1 (Further properties of eigenvalues/eigenspaces)
Throughout this exercise, A is a square matrix with entries in $\mathbb{R}$.

(a) Prove or disprove that A and A^t have the same eigenvalues. (A^t is the transpose of A, see lecture notes for Linear Algebra I.)

(b) Prove or disprove that A and A^t have the same eigenspaces.

(c) Assume A is regular and let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ. Show that $\mathbf{v}$ is also an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

(d) Let $\mathbf{v}$ be an eigenvector of the matrix A with eigenvalue λ and let s be a scalar. Show that $\mathbf{v}$ is an eigenvector of $A - sE$ with eigenvalue $\lambda - s$.

Solution:

a) The claim is correct: if λ is an eigenvalue of A, then $\det(A - \lambda E) = 0$. Since the determinant is invariant under transposition, also $\det(A-\lambda E)^t = \det(A - \lambda E) = 0$. Making the following computation: $(A - \lambda E)^t = A^t - (\lambda E)^t = A^t - \lambda E$, we see that $\det(A^t - \lambda E) = 0$. Therefore λ is also an eigenvalue of A^t. The other direction follows by symmetry ($(A^t)^t = A$).

b) This claim is incorrect, as can be seen by considering the following counterexample. When $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the only eigenvalue of A is 0. Its eigenspace is $V_0 = \text{span}(\mathbf{e}_1) = \{\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \alpha \in \mathbb{R}\}$. 0 is also the only eigenvalue of A^t, but its eigenspace is $V_0' = \text{span}(\mathbf{e}_2) = \{\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R}\}$.

c) Let λ be an eigenvalue of A and $\mathbf{v}$ be one of its eigenvectors. Then $A\mathbf{v} = \lambda\mathbf{v}$ holds. Since A is invertible, A^{-1} exists. By multiplying the previous equation by A^{-1} from the left, we obtain $\mathbf{v} = \lambda A^{-1}\mathbf{v}$. Now we multiply by $\frac{1}{\lambda}$ and get $A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$ ($\lambda \neq 0$ as A is regular). Thus $\mathbf{v}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

d) Combining $A\mathbf{v} = \lambda\mathbf{v}$ and $sE\mathbf{v} = s\mathbf{v}$ we get: $A\mathbf{v} - sE\mathbf{v} = \lambda\mathbf{v} - s\mathbf{v}$. This can be written as $(A - sE)\mathbf{v} = (\lambda - s)\mathbf{v}$, therefore $\mathbf{v}$ is an eigenvector of $A - sE$ with eigenvalue $\lambda - s$.

Exercise 2 (Application of diagonalisation: Fibonacci Numbers, Golden Mean)
Recall that the sequence $f_0, f_1, f_2, \ldots$ of Fibonacci numbers is inductively defined as follows (cf Exercise H6.4 from LA I):

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_{k+2} &= f_{k+1} + f_k. \end{aligned}$$

(a) We define $\mathbf{u}_k = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} \in \mathbb{R}^2$. Find a matrix A such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$ for all $k \in \mathbb{N}$.

(b) What are the eigenvalues of A? Give an explicit formula for f_k.
Hint: Use the eigenvalues λ_1 and λ_2 as abbreviations as long as possible.

(c) Compute the limit $a = \lim_{k\to\infty} \frac{f_{k+1}}{f_k}$.
The limit is called the Golden Mean, it divides a line segment of length 1 into two parts a and $1-a$ such that $\frac{1}{a} = \frac{a}{1-a}$.

Solution:

a) $\mathbf{u}_{k+1} = \begin{pmatrix} f_{k+2} \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} f_{k+1} + f_k \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{u}_k$. Therefore,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

As a consequence, for all $k \in \mathbb{N}$,

$$\mathbf{u}_k = A^k \mathbf{u}_0 = A^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

b) $\det(A - \lambda E) = \lambda^2 - \lambda - 1 = (\lambda - \frac{1+\sqrt{5}}{2})(\lambda - \frac{1-\sqrt{5}}{2}) = 0$.
Thus, the eigenvalues of A are:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ for λ_1 and $\mathbf{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$ for λ_2.

It follows that $A = SDS^{-1}$, with $S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. The conjugation map $M \mapsto SMS^{-1}$ is an automorphism of the ring of $n \times n$ matrices (and in particular preserves sums and products of matrices), so we have $A^k = (SDS^{-1})^k = SD^kS^{-1}$. Since $S^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$,

$$\begin{aligned} A^k &= SD^kS^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{k+1} & \lambda_2^{k+1} \\ \lambda_1^k & \lambda_2^k \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} & -\lambda_1^{k+1}\lambda_2 + \lambda_2^{k+1}\lambda_1 \\ \lambda_1^k - \lambda_2^k & -\lambda_1^k\lambda_2 + \lambda_2^k\lambda_1 \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = \mathbf{u}_k = A^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k \end{pmatrix},$$

hence

$$f_k = \frac{1}{\sqrt{5}}(\lambda_1^k - \lambda_2^k) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right).$$

c) We have that

$$\begin{aligned} \frac{f_{k+1}}{f_k} &= \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1^k - \lambda_2^k} = \frac{\lambda_1^{k+1}}{\lambda_1^k} \cdot \frac{1 - \frac{\lambda_2^{k+1}}{\lambda_1^{k+1}}}{1 - \frac{\lambda_2^k}{\lambda_1^k}} = \lambda_1 \cdot \frac{1 - (\frac{\lambda_2}{\lambda_1})^{k+1}}{1 - (\frac{\lambda_2}{\lambda_1})^k} \quad \text{and} \\ \left| \frac{\lambda_2}{\lambda_1} \right| &= \left| \frac{1-\sqrt{5}}{1+\sqrt{5}} \right| = \frac{\sqrt{5}-1}{\sqrt{5}+1} < 1. \end{aligned}$$

It follows that

$$\lim_{k\to\infty} \frac{f_{k+1}}{f_k} = \lambda_1 \cdot \frac{1 - \lim_{k\to\infty} \left(\frac{\lambda_2}{\lambda_1} \right)^{k+1}}{1 - \lim_{k\to\infty} \left(\frac{\lambda_2}{\lambda_1} \right)^k} = \lambda_1.$$

Exercise 3 (Euclidean algorithm and recursive functions)

The greatest common divisor $d = \gcd(a,b)$ of two non-zero natural numbers a and b is a natural number characterised by the following property:

- $d|b$ and $d|a$.
- If $r|a$ and $r|b$, then $r|d$

The Euclidean algorithm is a procedure for determining the greatest common divisor of two numbers a and b.

Step 0: Swap a and b if $a < b$.
Euclid (a,b): IF $b = 0$ THEN return a, ELSE return Euclid$(b, a \bmod b)$.

[After initialising: $d_1 := \min\{a,b\}, d_0 := \max\{a,b\}$, we divide with remainder in each step: $d_{k-1} = q_k d_k + d_{k+1}$ with $0 \le d_{k+1} < d_k$, and ends if $d_{k+1} = 0$. We get $d_k = \gcd(a,b)$.]

(a) Prove that the gcd is well defined, that is, uniquely characterised by the above properties.

(b) Prove that $\gcd(a,b) = \gcd(b, a \bmod b)$ for $0 < b$.

(c) Assume that $b < a$ and that $(a,b) \xrightarrow{E} (a',b') \xrightarrow{E} (a'',b'')$ are two steps in the Euclidean algorithm. Prove that $a'' < \frac{a}{2}$ and $b'' < \frac{b}{2}$.

(d) Deduce that Euclid(a,b) returns $\gcd(a,b)$.

Solution:

a) 1 divides both a and b so the set of common divisors is non-empty. Let d and d' satisfy the second condition. Then $d|d'$ and $d'|d$, so $d = d'$.

b) Suppose $a = qb + r$ where $0 \le r < b$, so $r = a \bmod b$. Since $e|a$ and $e|b$ *iff* $e|r = a - qb$ and $e|b$. Let $d = \gcd(a,b)$ and $d' = \gcd(b,r)$. We have $d|d'$ and $d'|d$, so $d = d'$.

c) If $b \le \frac{a}{2}$, then $b' < b \le \frac{a}{2}$; if $\frac{a}{2} < b$, then $q = 1$ so $b' = a - b < \frac{a}{2}$. So in every case, $b' < \frac{a}{2}$. Similarly $b'' < \frac{a'}{2}$. Moreover $a' = b$ and $a'' = b'$, so $a'' < \frac{a}{2}$ and $b'' < \frac{b}{2}$.

d) Every step $(a,b) \xrightarrow{E} (a',b')$ of the algorithm preserves the gcd, in the sense that $\gcd(a,b) = \gcd(a',b')$. Moreover the algorithm terminates due to part (c), so we obtain $(\gcd(a,b), 0)$ after finitely many steps.

Exercise 4 (Diagonalization and recursive sequences)

Let a_k be the sequence of real numbers defined recursively as follows: $a_0 = 0$, $a_1 = 1$, and $a_{k+2} = \frac{1}{2}(a_{k+1} + a_k)$. In other words, each term in the sequence is the average of the two previous terms.

(a) As we did with the Fibonacci sequence, we want to study this sequence using diagonalization of matrices. For $k \ge 0$, let

$$\mathbf{u}_k = \begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix}.$$

Using the equations $a_{k+2} = \frac{1}{2}(a_{k+1} + a_k)$ and $a_{k+1} = a_{k+1}$, find a 2×2 matrix A such that $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

(b) Find the eigenvalues and eigenvectors of A, and find a matrix S and a diagonal matrix D with $D = S^{-1}AS$.

(c) Find a formula for a_k, and calculate $\lim_{k\to\infty} a_k$ if it exists.

Solution:

a) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$.

b) The eigenvalues of A are 1 and $-\frac{1}{2}$.

For $\lambda = 1$, the eigenspace V_1 is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = -\frac{1}{2}$, the eigenspace $V_{-1/2}$ is spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. So $S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.

c) First, we have $\mathbf{u}_k = A^k\mathbf{u}_0 = SD^kS^{-1}\mathbf{u}_0$, where $\mathbf{u}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and a_k is just the second component of the vector $\mathbf{u}_k$. We calculate

$$S^{-1} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix}, \quad S^{-1}\mathbf{u}_0 = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}, \quad D^k = \begin{pmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{pmatrix}.$$

It follows easily that $a_k = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^k$. It is clear from this formula that $\lim_{k\to\infty} a_k = \frac{1}{3}$.