# Linear Algebra II
# Exercise Sheet no. 1

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**Exercise G1** (Warm-up)

In $\mathbb{R}^3$, let $g$ be a line through the origin and $E$ be a plane through the origin such that $g$ is not in $E$. Determine (geometrically) the eigenvalues and eigenspaces of the following linear maps:

(a) reflection in the plane $E$.

(b) central reflection in the origin.

(c) parallel projection in the direction of $g$ onto $E$.

(d) rotation about $g$ through $\frac{1}{3}\pi$ followed by rescaling in the direction of $g$ with factor 6.

Which of these maps admit a basis of eigenvectors?

**Solution:**

a) Two eigenvalues: 1, with eigenspace $E$, and $-1$, whose corresponding eigenspace is the orthogonal complement of $E$.

b) One eigenvalue $-1$ with eigenspace $\mathbb{R}^3$.

c) Two eigenvalues: 0, with eigenspace $g$, and 1, with eigenspace $E$.

d) One eigenvalue: 6, with eigenspace $g$.

We have a basis of eigenvectors in cases (i), (ii) and (iii).

**Exercise G2** (Warm-up)

(a) Suppose that $\varphi : V \to V$ is a linear map over an arbitrary field, and such that all vectors $\mathbf{v} \in \mathbf{V}$ are eigenvectors of $\varphi$. Show that $\varphi$ must have exactly one eigenvalue $\lambda$, and that $\varphi$ is precisely $\lambda \cdot \mathrm{id}$, where id is the identity map.

(b) Let $\psi : \mathbb{R}^4 \to \mathbb{R}^4$ be the map defined by

$$\varphi \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ -w \\ z \end{pmatrix}.$$

Find the (real) eigenvalues of $\varphi$ and their multiplicity, and find bases for the corresponding eigenspaces.

**Solution:**

a) Suppose that $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $\varphi$. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be (non-zero) vectors in the corresponding eigenspaces, so that $\varphi(\mathbf{v_1}) = \lambda_1\mathbf{v_1}$ and $\varphi(\mathbf{v_2}) = \lambda_2\mathbf{v_2}$. Clearly $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent. Then $\varphi(\mathbf{v_1} + \mathbf{v_2}) = \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2}$. Since every vector in $V$ is an eigenvector, this must be equal to $\lambda(\mathbf{v_1} + \mathbf{v_2})$ for some scalar $\lambda$. Then

$$(\lambda_1 - \lambda)\mathbf{v}_1 + (\lambda_2 - \lambda)\mathbf{v}_2 = \mathbf{0},$$

which implies that $\lambda_1 = \lambda = \lambda_2$, by linear independence. This is a contradiction, so there is only one eigenvalue $\lambda$. It is immediate that $\varphi = \lambda \cdot \mathrm{id}$.

b) The only eigenvalue is 1, and a basis for the corresponding eigenspace consists of $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

**Exercise G3** (Fixed points of affine maps)

Recall that an affine map is a function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\varphi(\mathbf{x}) = \varphi_0(\mathbf{x}) + \mathbf{b}$ where $\varphi_0$ is a linear map and $\mathbf{b} \in \mathbb{R}^2$ is a vector. In this exercise we are interested in the question of whether such a map $\varphi$ has a *fixed point,* i.e., a point $\mathbf{x}$ such that $\varphi(\mathbf{x}) = \mathbf{x}$.

(a) Prove that $\varphi$ has a fixed point, provided that 1 is not an eigenvalue of $\varphi_0$.

(b) Let $\varphi$ be a rotation through the angle $\alpha$ about a point $\mathbf{c}$. Give a formula for $\varphi$ w.r.t. the standard basis, i.e., find functions $f$ and $g$ such that $\varphi(x, y) = (f(x, y), g(x, y))$.

(c) Let $\varrho_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation through the angle $\alpha$ (about the origin) and let $\tau_{\mathbf{c}} : \mathbf{x} \mapsto \mathbf{x} + \mathbf{c}$ be the translation by $\mathbf{c}$. Using (ii), show that the composition $\tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}}$ is a rotation through $\alpha$ about the point $\mathbf{c}$.

(d) Suppose that the linear map $\varphi_0$ is a rotation through an angle $\alpha \neq 0$. Prove that the affine map $\varphi : \mathbf{x} \mapsto \varphi_0(\mathbf{x}) + \mathbf{b}$ has a fixed point $\mathbf{c}$ and that $\varphi = \tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}}$, i.e., $\varphi$ is a rotation through $\alpha$ about $\mathbf{c}$.

(Bonus question: how can you find the centre $\mathbf{c}$ *geometrically* (i.e., without computation)?)

(e) Give an example of an affine map $\varphi(\mathbf{x}) = \varphi_0(\mathbf{x}) + \mathbf{b}$ without fixed points such that $\varphi_0$ is not the identity map.

**Solution:**

a) A point $\mathbf{x}$ is a fixed point of $\varphi$ if $\varphi_0(\mathbf{x}) + \mathbf{b} = \mathbf{x}$. We can rewrite this equation as

$$(\varphi_0 - \mathrm{id}\,)(\mathbf{x}) = -\mathbf{b}\,.$$

We claim that the map $\varphi_0 - \mathrm{id}$ is invertible. Since we are in a finite dimensional vector space it is sufficient to show that $\varphi_0 - \mathrm{id}$ is injective, i.e., that $\ker(\varphi_0 - \mathrm{id}\,) = 0$. Suppose that $\mathbf{v} \in \ker(\varphi_0 - \mathrm{id}\,)$. Then $0 = (\varphi_0 - \mathrm{id}\,)(\mathbf{v}) = \varphi_0(\mathbf{v}) - \mathbf{v}$. Hence, $\varphi_0(\mathbf{v}) = \mathbf{v}$. Since $\varphi_0$ does not have the eigenvalue 1, the only solution to this equation is $\mathbf{v} = \mathbf{0}$. Hence, $\ker(\varphi_0 - \mathrm{id}\,) = 0$, as desired.

It follows that $\varphi_0 - \mathrm{id}$ is invertible and

$$\mathbf{x} = (\varphi_0 - \mathrm{id}\,)^{-1}(-\mathbf{b})$$

is a fixed point.

b) The rotation through $\alpha$ around the origin is the function

$$\varrho_\alpha(x, y) = \big(x \cos\alpha - y \sin\alpha,\ x \sin\alpha + y \cos\alpha\big)\,.$$

If we rotate around $\mathbf{c} = (c_x, c_y)$, we obtain

$$\varphi(x, y) = \big(c_x + (x - c_x)\cos\alpha - (y - c_y)\sin\alpha,\ c_y + (x - c_x)\sin\alpha + (y - c_y)\cos\alpha\big)\,.$$

c) Direct computation shows that

$$\begin{aligned}(\tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}})(x, y) = \big(&c_x + (x - c_x)\cos\alpha - (y - c_y)\sin\alpha,\\ &c_y + (x - c_x)\sin\alpha + (y - c_y)\cos\alpha\big)\,.\end{aligned}$$

Hence, the result follows by (ii).

d) To show that $\varphi$ has a fixed point $\mathbf{c}$ it is sufficient, by (i), to show that $\varphi_0$ does not have the eigenvalue 1. Hence, suppose that $\mathbf{v}$ is a vector with $\varphi_0(\mathbf{v}) = \mathbf{v}$. Since $\varphi_0$ is a rotation through an angle $\alpha \neq 0$ it only fixes the zero vector. Hence, $\mathbf{v} = \mathbf{0}$ is the only solution to this equation.

It remains to prove that $\varphi = \tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}}$. Note that $\varphi_0 = \varrho_\alpha$. Hence, we have

$$\begin{aligned}(\tau_{\mathbf{c}} \circ \varrho_\alpha \circ \tau_{-\mathbf{c}})(\mathbf{x}) &= \varrho_\alpha(\mathbf{x} - \mathbf{c}) + \mathbf{c}\\ &= \varrho_\alpha(\mathbf{x}) - \varrho_\alpha(\mathbf{c}) + \mathbf{c}\\ &= \varrho_\alpha(\mathbf{x}) - \varrho_\alpha(\mathbf{c}) - \mathbf{b} + \mathbf{c} + \mathbf{b}\\ &= \varrho_\alpha(\mathbf{x}) - \varphi(\mathbf{c}) + \mathbf{c} + \mathbf{b}\\ &= \varrho_\alpha(\mathbf{x}) + \mathbf{b}\\ &= \varphi(\mathbf{x})\,.\end{aligned}$$

e) For instance, the map

$$\varphi(x,y) = (-x, 1+x+y)$$

has no fixed point. (It must have eigenvalue 1. What is the corresponding eigenvector?)

**Exercise G4** (Eigenvalues and eigenvectors)

Consider the real $2\times 2$ matrix $A = \begin{pmatrix} -2 & 6 \\ -2 & 5 \end{pmatrix}$ and the linear map $\varphi = \varphi_A$ given by $A$ w.r.t. the standard basis.

(a) Calculate the eigenvalues of $A$ by expanding $\det(A-\lambda E)$ and find the zeroes/roots of the characteristic polynomial.

(b) For each eigenvalue $\lambda_i$ determine the eigenspace $V_{\lambda_i}$.

(c) Find a basis $B$ of $\mathbb{R}^2$ that only consists of eigenvectors of $\varphi$ and find the matrix of the map $\varphi$ with respect to the basis $B$.

**Solution:**

a) We have

$$det(A-\lambda E) = (-2-\lambda)(5-\lambda)+12 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)\,.$$

Thus the characteristic polynomial splits into linear factors corresponding to roots $\lambda_1 = 1$ and $\lambda_2 = 2$.

b) In order to determine the kernels of $A - \lambda_i E$, we perform Gauss–Jordan elimination.

$$A-\lambda_1 E = \begin{pmatrix} -3 & 6 \\ -2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -3 & 6 \\ 0 & 0 \end{pmatrix}$$
$$A-\lambda_2 E = \begin{pmatrix} -4 & 6 \\ -2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -4 & 6 \\ 0 & 0 \end{pmatrix}$$

We may choose $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ such that $\mathrm{span}(\mathbf{v}_1) = \ker(A-\lambda_1 E)$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ with $\mathrm{span}(\mathbf{v}_2) = \ker(A-\lambda_2 E)$.

c) The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis $B$ of $\mathbb{R}^2$, since they are linearly independent. W.r.t. to this basis $\varphi$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.