# Script Skeleton: Advanced Complexity Theory*

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**Abstract.** Classical computational complexity theory (last semester) has resulted in a rich variety of classes naturally capturing many practical computational problems. We also have established several relations between these classes; but not a single non-trivial lower bound (other than, e.g. $\mathrm{Time}(n) \geq n$ and $\mathrm{Space}(n) \geq \log n$) was obtained. The topics of this lecture describe several (more or less successful) approaches to remedy this situation.



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# 1 Algorithmic Information Theory and Applications

0101010101010101 vs. 1011001011111001 Here $\Sigma := \{0, 1\}$.

**Definition 1.1.** *Fix a universal Turing machine $\mathcal{U}$ over alphabet $\Sigma$ and let, for $\bar{x} \in \Sigma^*$, $K_\mathcal{U}(\bar{x}) := \min\big\{|\langle\mathcal{M}, \bar{y}\rangle| : \mathcal{U}(\langle\mathcal{M}\rangle, \bar{y}) = \bar{x}\big\}$.*

**Lemma 1.2 (Kolmogorov Complexity).**

*a) There exists $c \in \mathbb{N}$ such that, for every $\bar{x} \in \Sigma^*$, $K_\mathcal{U}(\bar{x}) \leq c + |\bar{x}|$.*
*b) To UTM $\mathcal{V}$ there exists $c \in \mathbb{N}$ such that every $\bar{x} \in \Sigma^*$ has $K_\mathcal{U}(\bar{x}) \leq c + K_\mathcal{V}(\bar{x})$*
*c) To every $n$, there exists $\bar{x} \in \Sigma^n$ with $K_\mathcal{U}(\bar{x}) \geq n$.*

Strings $\bar{x}$ with $K(\bar{x}) \approx |\bar{x}|$ are considered *incompressible.*

## 1.1 A Lower Bound for 1-Tape Turing Machines

**Definition 1.3 (Crossing Sequence).** *Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta)$ denote a deterministic 1-tape Turing machine (1-DTM) $\bar{\in}\Sigma^*$, and $s \in \mathbb{N}_0$. Then $\mathrm{CS}_\mathcal{M}(\bar{x}, s)$ denotes the finite or infinite sequence $(q_i)$ of states $\mathcal{M}$ is in when moving from tape cell #s-1 to #s or back. We write $|\,\mathrm{CS}_\mathcal{M}(\bar{x}, s)|$ for its length.*

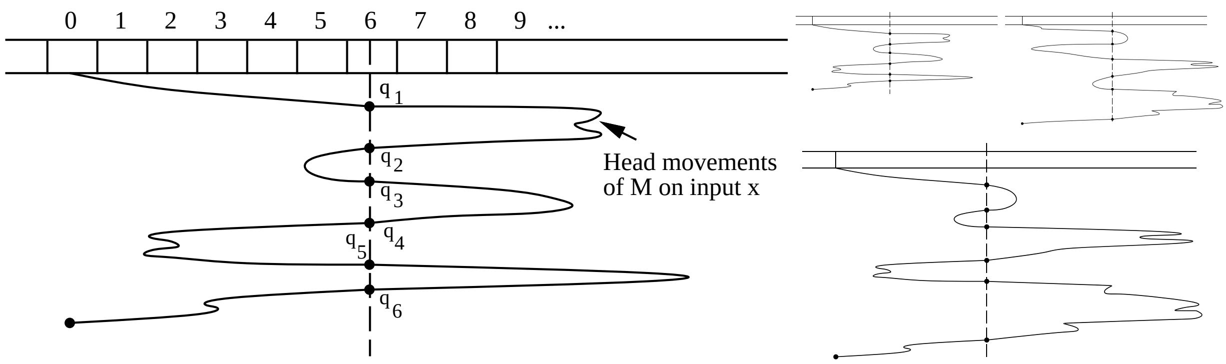


**Fig. 1.** a) Example of a Crossing Sequence b) Combining two computations with identical crossing sequences

**Lemma 1.4 (Pumping).** *Let $\mathcal{M}$ as above.*

*a) Suppose both $\bar{u}\bar{v}$ and $\bar{y}\bar{z}$ are accepted by $\mathcal{M}$ and satisfy $\mathrm{CS}_\mathcal{M}(\bar{u}\bar{v}, |\bar{u}|) = \mathrm{CS}_\mathcal{M}(\bar{y}\bar{z}, |\bar{y}|)$. Then $\mathcal{M}$ accepts also $\bar{u}\bar{z}$ and $\bar{y}\bar{v}$.*
*b) $\mathrm{Time}_\mathcal{M}(\bar{x}) = \sum_{s=1}^{\infty} |\,\mathrm{CS}_\mathcal{M}(\bar{x}, s)|$. In particular, to every finite $S \subseteq \mathbb{N}_0$, there exists $s \in S$ with $|\,\mathrm{CS}_\mathcal{M}(\bar{x}, s)| \leq \mathrm{Time}_\mathcal{M}(\bar{x})/|S|$.*
*c) Suppose $\mathcal{M}$ decides $L := \big\{\bar{x}\ 0^{|\bar{x}|}\ \bar{x} : \bar{x} \in \Sigma^*\big\}$. Then, for $m \leq s < 2m$ and $\bar{x}, \bar{y} \in \Sigma^m$ with $\bar{x} \neq \bar{y}$, it holds $\mathrm{CS}_\mathcal{M}(\bar{x}0^m\bar{x}, s) \neq \mathrm{CS}_\mathcal{M}(\bar{y}0^m\bar{y}, s)$.*
*d) There is $c \in \mathbb{N}$ such that every $\bar{x} \in \Sigma^m$ has $K_\mathcal{U}(\bar{x}) \leq c + c \cdot \mathrm{Time}_\mathcal{M}(\bar{x}0^m\bar{x})/m + \log_2(m)$.*

**Theorem 1.5.** *The language L from Lemma 1.4c)*

*a) can be decided by a deterministic 2-DTM in time* $\mathcal{O}(n)$
*b) can be decided by a deterministic 1-DTM in time* $\mathcal{O}(n^2)$
*c) cannot be decided by a deterministic 1-DTM in time* $o(n^2)$

# 2 Time versus Space

For every $f(n) \geq n$,

a) $\text{DTIME}\big(f(n)\big) \subseteq \text{DSPACE}\big(f(n)\big)$
b) $\text{DSPACE}\big(f(n)\big) \subseteq \text{DTIME}\big(2^{\mathcal{O}(f(n))}\big)$.

Improving b): $\mathcal{P}$ versus $\mathcal{NP}$. This section improves a).
$\text{DTIME}_k\big(f(n)\big)$ depends on the number $k$ of heads; $\text{DSPACE}_k\big(f(n)\big)$ does not.

## 2.1 Pebble Game: Time versus Space for Circuits

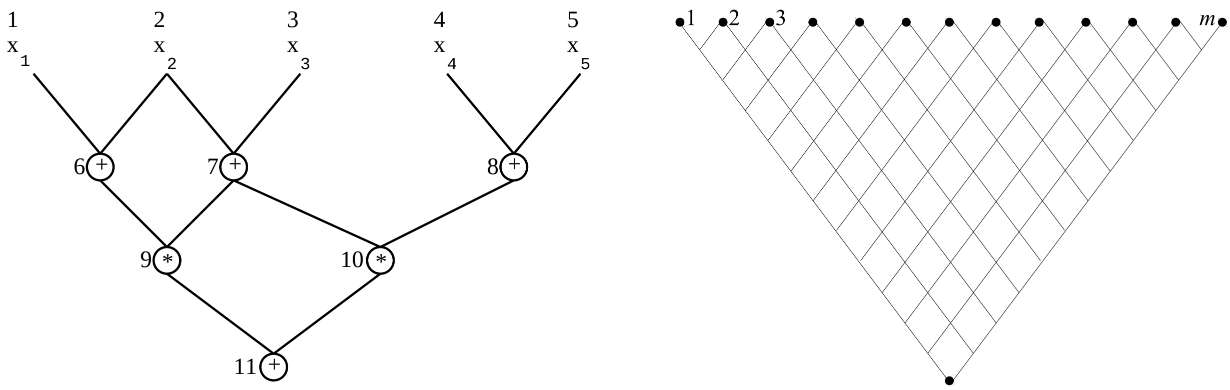


**Fig. 2.** a) An arithmetic circuit b) Pyramid graph $P_m$

**Definition 2.1 (Pebble Game).** *Let* $G = (V, E)$ *denote a directed acyclic graph (DAG) and* $v \in V$. *The goal of the game instance* $(V, E, v)$ *is to eventually put a marker ('*pebble*') on* $v$, *by a sequence of moves subject to the following rules:*

- *Either remove a marker from some vertex* $u \in V$
- *Or put a marker onto a vertex* $u \in V$, provided *that all direct predecessors of* $u$ *are presently marked.*

*We count the number of steps as well as the number of (reusable) markers employed.*

**Example 2.2** *The DAG in Figure 2a) can be played*

*a) with 5 markers in 17 steps*
*b) but not with 4 markers.*

*For $m \in \mathbb{N}$, the pyramid graph $P_m$ from Figure 2b) can be played*

*c) with $m+1$ markers*
*d) but not with $m$ markers* □

**Lemma 2.3.** *Let $(V,E)$ denote a DAG with indegree at most $\ell \in \mathbb{N}$ and $m := |E|$ edges. Write $S_\ell(m)$ for the least number of markers sufficient to play* every *DAG of indegree $\leq \ell$ having $\leq m$ edges.*

*a) There exists $U \subseteq V$ such that $F := E \cap (U \times U)$ has $m/2 - \ell \leq |F| < m/2$ and $E \cap \big((V \setminus U) \times U\big) = \emptyset$.*
*b) It holds $S_\ell(m) \leq S_\ell(m/2+\ell) + |F'|$ with the abbreviation $F' := E \cap \big(U \times (V \setminus U)\big)$.*
*c) It also holds $S_\ell(m) \leq S_\ell(m/2) + S_\ell(m/2+\ell-|F'|) + \ell$.*
*d) It holds $S_\ell(m) \leq \max\big\{S_\ell(m/2+\ell) + \frac{2m}{\log m}, S_\ell(m/2) + S_\ell(m/2+\ell-\frac{2m}{\log m}) + \ell\big\}$ and, for fixed $\ell$, $S_\ell(m) \leq \mathcal{O}(m/\log m)$.*
*e) $(V,E)$ can be played with $\mathcal{O}(n/\log n)$ markers for $n := |V|$ and $\ell$ considered fixed.*

### 2.2 Pebble Strategies and Computation

We encode a DAG $(V,E)$ as a list of vertices and edges, i.e. such that $N := |\langle V,E\rangle| = \Theta(n \cdot \log n + m \cdot \log n)$ where $n := |V|$ and $m := |E|$.

**Lemma 2.4.** *a) A quadratically space-bounded DTM can, given $\langle V,E,v,s,t\rangle$, produce some play for $(V,E,v)$ using $\leq s$ markers and $\leq 2^t$ steps, provided that such a play exists.*

*b) Let $\mathcal{M}$ denote a $k$-tape Turing machine and $\bar{x} \in \Sigma^n$ an input on which $\mathcal{M}$ makes $T$ steps. Subdivide this computation into $B \in \mathbb{N}$ phases of $\lceil T/B \rceil$ steps each; and subdivide $\mathcal{M}$'s tapes into $B$ blocks of $\lceil T/B \rceil$ cells each.*
  *i) In each phase and on each tape, $\mathcal{M}$ visits at most 2 different blocks.*
  *ii) The computation of $\mathcal{M}(\bar{x})$ in phase $\varphi = 1, \ldots, B$ depends on the contents of blocks last modified in at most $\ell := k+1$ different previous phases.*

*c) Choosing $B := \lceil T^{1/3} \rceil$, the computation of $\mathcal{M}$ on $\bar{x}$ can be simulated by a Turing machine using space $\mathcal{O}(T/\log T)$.*

**Theorem 2.5 (Hopcroft, Paul, Valiant 1977).** *Let $t(n) \geq n$ be constructible in space $t(n)/\log t(n)$. Then* $\mathrm{DTIME}\big(t(n)\big) \subseteq \mathrm{DSPACE}\big(t(n)/\log t(n)\big)$.

## 3 Simple Diagonalization: Hartmanis' Hierarchy Theorems

**Example 3.1** *The following language is not semi-decidable:*

$$D \quad := \quad \{\langle\mathcal{M}\rangle : \textit{DTM } \mathcal{M} \textit{ does not accept } \langle\mathcal{M}\,\rangle\} \quad \subseteq \quad \Sigma^*$$

We consider DTMs over arbitrary finite alphabets.

**Proposition 3.2.** *Fix $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \geq 2n$.*

*a) It holds $T_f \notin \mathrm{DTIME}\,\big(f(n)\big)$, where*

$$T_f \quad := \quad \big\{\langle\mathcal{M}\rangle : \textit{ DTM } \mathcal{M} \textit{ does not accept } \langle\mathcal{M}\rangle \textit{ within } f(|\langle\mathcal{M}\rangle|) \textit{ steps } \big\}$$

*b) It also holds $S_f \notin \mathrm{DSPACE}\,\big(f(n)\big)$ for the language*

$$S_f \quad := \quad \big\{\langle\mathcal{M}\rangle : \textit{ DTM } \mathcal{M} \textit{ does not accept } \langle\mathcal{M}\rangle \textit{ using} \leq f(|\langle\mathcal{M}\rangle|) \textit{ tape cells } \big\}$$

*c) If $f$ is computable in space $\mathcal{O}\big(f(n)\cdot\log n\big)$, then $S_f \in \mathrm{DSPACE}\,\big(f(n)\cdot\log f(n)\big)$.*
*d) If $f$ is computable in time $\mathcal{O}\big(f(n)^3\big)$, then $T_f \in \mathrm{DTIME}\,\big(f(n)^3\big)$.*

**Corollary 3.3.** $\mathcal{P} \subsetneq \mathsf{EXP}$, $\mathcal{L} \subsetneq \mathsf{PSPACE}$.

**Theorem 3.4 (Hartmanis et. al., Fürer, Trakhtenbrot).**

*a) Let $f(n) \in o\big(g(n)\big)$ be computable in space $\mathcal{O}\big(f(n)\big)$. Then $\mathrm{DSPACE}\,\big(f(n)\big) \subsetneq \mathrm{DSPACE}\,\big(g(n)\big)$.*
*b) Let $f(n) \in o\big(g(n)\big)$ be computable in time $\mathcal{O}\big(f(n)\big)$. Then $\mathrm{DTIME}\,\big(f(n)\big) \subsetneq \mathrm{DTIME}\,\big(g(n)\big)$.*
*c) There is a computable monotone function $f(n) \geq n$ such that $\mathrm{DTIME}\,\big(f(n)\big) = \mathrm{DTIME}\,\big(2^{f(n)}\big)$.*

## 4 Relativization and Priority/Injury-Diagonalization: Baker/Gill/Solovay and Friedberg/Muchnik

**Example 4.1** *a) The language*

$$H \quad := \quad \{\langle\mathcal{M}, x\rangle : \textit{DTM } \mathcal{M} \textit{ terminates on input } x\}$$

*is i) semi-decidable but ii) not decidable. Moreover iii) every semi-decidable problem is many-one reducible to $H$.*

*b) $H$ is trivially decidable by an ODTM with oracle $H$; and so is $D$. But w.r.t. any oracle $O$, the following language is not semi-decidable by an $O$-oracle machine:*

$$D^O \quad := \quad \{\langle\mathcal{M}^?\rangle : \textit{ODTM } \mathcal{M}^O \textit{ does not accept } \langle\mathcal{M}^?\rangle\} \quad \subseteq \quad \Sigma^*$$

*c) There is a countably infinite hierarchy $\emptyset, H, H^H =: H', H^{H^H} =: H'', \ldots$ of languages; each $H^{(j)}$ semidecidable, but not decidable, relative to $H^{(j-1)}$.*

For complexity class $\mathcal{C}$ and oracle $O$, generically understand $\mathcal{C}^O$ to denote its relativization.

**Scholium 4.2** *For any oracle $O \subseteq \Sigma^*$, the following holds:*

*a) For increasing $f : \mathbb{N} \to \mathbb{N}$ $O$-computable in time and space $\mathcal{O}\big(f(n)\big)$,*

$$\mathrm{DTIME}\,\big(f(n)\big) \;\subseteq\; NTIME\big(f(n)\big) \;\subseteq\; DSPACE\big(f(n)\big) \;\subseteq\; DTIME\big(\mathcal{O}(2)^{\log n + f(n)}\big)$$

*b) If $f(n) \in o\big(g(n)\big)$ is computable in space $\mathcal{O}\big(f(n)\big)$, then $\mathrm{DSPACE}\,\big(f(n)\big) \subsetneq \mathrm{DSPACE}\,\big(g(n)\big)$.*
*If $f(n) \in o\big(g(n)\big)$ is computable in time $\mathcal{O}\big(f(n)\big)$, then $\mathrm{DTIME}\,\big(f(n)\big) \subsetneq \mathrm{DTIME}\,\big(g(n)\big)$.*

*c) If $s : \mathbb{N} \to \mathbb{N}$ with $s(n) \geq \log(n)$ is $O$-computable in space $\mathcal{O}\big(s(n)\big)$, then $\mathrm{NSPACE}^O\,\big(s(n)\big) \subseteq \mathrm{DSPACE}^O\,\big(s(n)^2\big)$.*

*d) For $s : \mathbb{N} \to \mathbb{N}$ with $s(n) \geq \log(n)$, it holds $\mathrm{NSPACE}^O\,\big(s(n)\big) = co\,\mathrm{NSPACE}^O\,\big(s(n)\big)$.*

*e) $\mathsf{BPP}^O \subseteq \Sigma_2\mathcal{P}^O \cap \Pi_2\mathcal{P}^O$.*

In particular, $\mathcal{L}^O \subseteq \mathcal{NL}^O \subseteq \mathcal{P}^O \subseteq \mathcal{NP}^O \subseteq \mathsf{PSPACE}^O = \mathsf{NPSPACE}^O \subseteq \mathsf{EXP}^O$ and at least one inclusion is strict.

**Theorem 4.3 (Baker, Gill, Solovay 1975).**

*a) There exists $A \subseteq \{0,1\}^*$ such that $\mathcal{P}^A = \mathcal{NP}^A$.*

*b) There exists an oracle $B \subseteq \{0,1\}^*$ such that $\mathcal{P}^B \neq \mathcal{NP}^B$.*

**Theorem 4.4 (Friedberg 1957/Muchnik 1956).** *There exist semidecidable $A, B \subseteq \{0,1\}^*$ such that $A$ is not decidable relative to $B$ and $B$ is not decidable relative to $A$.*

## 5 Straight-Line Complexity

**Definition 5.1 (Straight-Line Program).** *Let $\mathcal{S} = \big(S, (c_i), (f_j)\big)$ denote a structure with constants $c_i \in S$ and functions $f_j : S^{a_j} \to S$ of arities $a_j \in \mathbb{N}$. A* Straight-Line Program *$P_\mathcal{S}$ (over this structure and in variables $X_1, \ldots, X_n$) is a finite sequence of assignments $Z_k := c_i$ and $Z_k := X_\ell$ ($1 \leq \ell \leq n$) and $Z_k := f_j(Z_{k_1}, \ldots, Z_{k_{a_j}})$, $1 \leq k_1, \ldots, k_{a_j} < k$. When assigned values $x_1, \ldots, x_n \in S$ to $X_1, \ldots, X_n$, the program* computes *(the set of results consisting of $(x_1, \ldots, x_n) =: \boldsymbol{x}$ and of) $Z_1, \ldots, Z_K$; the final result is $Z_K =: P_\mathcal{S}(\boldsymbol{x})$. However if some intermediate operation $f_j(Z_{k_1}, \ldots, Z_{k_{a_j}})$ is undefined, then so is $P_\mathcal{S}(\boldsymbol{x}) := \perp$. A* cost function *$C$ assigns to each $f_j$ some cost $C(f_j) \geq 0$. The cost of a straight-line program $P$ is the sum of the costs of the $f_j$ occurring. The* length *of a straight-line program means its cost with respect to constant cost function $f_j \mapsto 1$.*

**Example 5.2** *a) Let $\mathcal{S} := \big(R, R, (+, \times)\big)$ be a commutative ring and $p \in R[X]$ a polynomial. Horner's Scheme gives rise to a straight-line program computing $x \mapsto p(x)$ of length at most $2 \cdot \deg(p)$.*

*b) Consider the semi-group $\mathcal{S} := \big(\mathbb{N}, (1), (+)\big)$. Every $N \in \mathbb{N}$ can be computed by a straight-line program over $\mathcal{S}$ of length at most $2 \cdot \lfloor \log_2 N \rfloor$.*

c) *Consider the $N$-dimensional discrete Fourier-transform*

$$\mathcal{F}_N : \mathbb{C}^N \ni (x_0, \ldots, x_{N-1}) \mapsto \Big( \sum\nolimits_{\ell=0}^{N-1} \exp(2\pi i \cdot k \cdot \ell / N) \cdot x_\ell \Big)_{k=0,\ldots,N-1} \in \mathbb{C}^N .$$

*For $N = 2^n$, $\mathcal{F}_N$ can be computed by a straight-line program over $\big(\mathbb{C}, (0), (+, \times_c : c \in \mathbb{S}^1)\big)$ of length $\mathcal{O}(N \cdot \log N)$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ denotes the complex unit circle and $\times_c : \mathbb{C} \to \mathbb{C}, z \mapsto c \cdot z$ unary complex multiplication by $c$.*

d) *Let $F$ denote a field and $\mathcal{A}$ an $F$-algebra. There is a straight-line program over $\big(\mathcal{A}, (), (+, \times_c : c \in F)\big)$ which, for arbitrary but fixed distinct $x_1, \ldots, x_n \in F$ and on input of $y_1, \ldots, y_n \in \mathcal{A}$, calculates (the unique) $a_0, \ldots, a_{n-1} \in \mathcal{A}$ with $\sum_{k=0}^{n-1} a_k \cdot x_\ell^k = y_\ell$ for $\ell = 1, \ldots, n$.*

e) *Consider an infinite field $F$, $n + m$ variables $A_0, \ldots, A_{n-1}, B_0, \ldots, B_{m-1}$, and the algebra $\mathcal{A} = F[A_0, \ldots, B_{m-1}]$ with binary operations $+$ and $\times : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ as well as unary $\times_c : \mathcal{A} \to \mathcal{A}$ ($c \in F$), that is the structure $\big(\mathcal{A}, F, (+, \times, \times_c : c \in F)\big)$. The set $\big\{ \sum_{i+j=\ell} A_i \cdot B_j : 0 \le \ell \le n+m-2 \big\}$ can be calculated from $A_0, \ldots, Y_{m-1}$ by a straight-line program using $n + m - 1$ operations "$\times$" (and arbitrary many "$+$" and "$\times_c$").*

Straight-line programs as in b) are *addition chains*; c) refers to the *fast Fourier transform.*

### 5.1 Lower Bounds: Dimension, Volume, Transcendence Degree

**Proposition 5.3.** *a) Any straight-line program computing $N \in \mathbb{N}$ over $\big(\mathbb{N}, 1, (+, -)\big)$ has length at least $\log_2 N$. In particular the straight-line program from Example 5.2b) is optimal up to a constant factor.*

*b) For $\mathcal{S} = \big(F, F, (+, -, \times)\big)$ a field of characteristic 0 and $0 \neq p \in F[X]$, any straight-line program computing $x \mapsto p(x)$ over $\mathcal{S}$ contains at least $\log_2 \deg(p)$ multiplications.*

*Let $F$ denote a field and $F(X)$ the field of univariate rational functions.*
*For coprime $p(X), q(X) \in F[X]$ define $\deg\big(p(X)/q(X)\big) := \max\big\{\deg(p), \deg(q)\big\}$.*

c) *Every $r(X) \in F(X)$ can be calculated by a straight-line program over $\big(F, F, (+, \times, \div)\big)$ of length at most $4\deg(r) + 1$.*

d) *Conversely, any straight-line program over $\big(F, F, (+, \times, \div)\big)$ computing $r(X) \in F(X)$ has length at least $\log_2 \deg(r)$.*

**Theorem 5.4 (Dimension Bound).** *Let $F \subseteq E$ denote fields and consider $x_1, \ldots, x_n, y_1, \ldots, y_m \in E$ and the induced $F$-vector spaces $X := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \in F\}$ and $Y := \{\mu_1 y_1 + \cdots + \mu_m y_m : \mu_j \in F\}$. Moreover consider the structure $\mathcal{S} = \big(E, F, (+, -, \times, \times_\lambda : \lambda \in F)\big)$ where $\times : E \times E \to E$ and $\times_\lambda : E \to E$, $e \mapsto \lambda \cdot e$.*

a) *Any straight-line program over $\mathcal{S}$ computing $\{y_1, \ldots, y_m\}$ from $(x_1, \ldots, x_n)$ contains at least $\dim_F(X + Y + F) - \dim_F(X + F)$ multiplications "$\times$".*

b) *The straight-line program from Example 5.2e) is optimal.*

**Theorem 5.5 (Morgenstern's Volume Bound).** *Fix $C>0$ and consider a straight-line program $P$ over the structure $\big(\mathbb{C},\mathbb{C},(+,\times_\lambda:|\lambda|\leq C)\big)$ in $n$ variables.*

*a) Each 'line' $\ell$ of $P$ computes an affine linear function $\varphi_\ell:\mathbb{C}^n\to\mathbb{C}$; and $P$ computes an affine linear map $\Phi_P:\mathbb{C}^n\ni\boldsymbol{x}\mapsto A_P\cdot\boldsymbol{x}+\boldsymbol{b}\in\mathbb{C}^{n+|P|}$, where $|P|$ denotes the length of $P$ and the first $n$ components are the identity.*

*b) For $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\in\mathbb{C}^n$ with $m\geq n$ write*

$$\Delta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)\ :=\ \max\big\{|\det(\boldsymbol{a}_{j_1},\ldots,\boldsymbol{a}_{j_n})|:1\leq j_1,\ldots,j_n\leq m\big\}\ .$$

*Then, for $1\leq k,\ell\leq m$ and $\lambda\in\mathbb{C}$, it holds $\Delta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,\lambda\cdot\boldsymbol{a}_k)\leq|\lambda|\Delta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$ and $\Delta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,\boldsymbol{a}_k+\boldsymbol{a}_\ell)\leq 2\Delta(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$.*

*c) The homogeneous linear map $A_P:\mathbb{C}^n\to\mathbb{C}^{n+|P|}$ from a) satisfies $\Delta(A_P)\leq(2C)^{|P|}$.*

*d) The matrix $\big(\exp(2\pi i\cdot k\cdot\ell/n)\big)_{0\leq k,\ell<n}$ has determinant of absolute value $n^{n/2}$.*

*e) The straight-line program from Example 5.2c) is asymptotically optimal.*

**Theorem 5.6 (Transcendence Degree Bound, Motzkin+Belaga).** *Let $F\subseteq E$ denote fields of characteristic 0 and $\mathcal{F}\subseteq E(\boldsymbol{X})$ a finite set of rational functions in indeterminates $(X_1,\ldots,X_n)=\boldsymbol{X}$. For $p_j,q_j\in E[\boldsymbol{X}]$ coprime over $F$ and $q_j$ monic, define $\mathrm{Coeff}_F(p_1/q_1,\ldots,p_m/q_m)$ as the field over $F$ generated by the coefficients from $p_1,\ldots,q_m$.*

*a) $\mathrm{Coeff}_F(\mathcal{F})$ is well-defined and coincides with $F\big(\{f(\boldsymbol{x}):\boldsymbol{x}\in F^n,f\in\mathcal{F}\}\big)$.*

*b) For $a_j,b_j,c_j,w_j\in E[\boldsymbol{X}]$ with $b_j\neq0$, $\mathrm{Coeff}_F(w_j{+}c_j{\cdot}a_j/b_j:j)\subseteq\mathrm{Coeff}_F(w_j,c_j,a_j,b_j:j)$.*

*c) Consider the structure $\mathcal{S}'=\big(E,F,(E,+,\times,\div)\big)$. Any straight-line program computing $\mathcal{F}$ over $\mathcal{S}'$ contains at least $\mathrm{trdeg}_F\big(\mathrm{Coeff}_F(\mathcal{F})\big)$ constants from $E$.*

*d) Consider a straight-line program $P$ over $\mathcal{S}:=\big(E,E,(+,\times,\div)\big)$ computing (intermediate) results $f_1,\ldots,f_N$.*

*i) There exist $0\neq b_j,a_j\in E[\boldsymbol{X}]$, $c_j\in E$ (j=1,…,N) such that $f_j=c_j\cdot a_j/b_j$ and $\mathrm{trdeg}_F\big(\mathrm{Coeff}_F(a_1,\ldots,b_N)\big)$ is at most the number of additions in $P$.*

*ii) There exist $0\neq v_j,u_j\in E[\boldsymbol{X}]$, $w_j\in E$ (j=1,…,N) such that $f_j=w_j+u_j/v_j$ and $\mathrm{trdeg}_F\big(\mathrm{Coeff}_F(u_1,\ldots,v_N)\big)$ is at most twice $P$'s number of multiplications/divisions.*

*e) Any straight-line program computing $\mathcal{F}$ over $\mathcal{S}$ contains at least $\mathrm{trdeg}_F\big(\mathrm{Coeff}_F(\mathcal{F})\big)-|\mathcal{F}|$ additions and $\big(\mathrm{trdeg}_F\big(\mathrm{Coeff}_F(\mathcal{F})\big)-|\mathcal{F}|\big)/2$ multiplications.*

## 5.2 Some Surprisingly Efficient Algorithms: Preconditioning, Baur-Strassen, Multipoint Evaluation

**Proposition 5.7 (Horner is not optimal with preconditioning).**
*Let $E$ denote a field and $f=\sum_{j=0}^n\alpha_jX^j\in E[X]$ a polynomial of degree $n$.*

*a) Suppose $f=(X^2-\xi)\cdot f_1(X)+\eta\in E[X]$ with $\xi,\eta\in E$. Then $f$ can be calculated from $X,X^2,\xi,\eta,f_1(X)$ (the latter of degree $n-2$) using 1 multiplication and 2 additions/subtractions.*

*b) Suppose that $h:=\sum_{2\ell+1\leq n}\alpha_{2\ell+1}X^\ell$ is either constant or a product of linear factors in $E[X]$. Then there is a straight-line program computing $f$ in $E[X]$ from $X$ and $X^2$ and some elements from $E$ using at most $\lfloor n/2\rfloor+2$ multiplications and $n$ additions/subtractions.*

c) *Suppose $E$ is algebraically closed (or real closed). Then there is a straight-line program computing $f$ in $E[X]$ from $X$ and some elements of $E$ using at most $\lfloor n/2\rfloor+3$ multiplications and $n+1$ additions/subtractions; and for $\alpha_0,\ldots,\alpha_n$ algebraically independent, this is optimal up to an additive constant.*

**Theorem 5.8 (Baur-Strassen).** *Fix a field $F$ of characteristic 0, $0,1\in C\subseteq F$, and let $P$ denote a straight-line program in $n$ variables over $\mathcal{S}=\big(F,C,(+,-,\times,\div)\big)$ computing $f\in F(X_1,\ldots,X_n)$.*
*Then there exists a straight-line program $P'$ in $n$ variables over $\mathcal{S}$ of length $|P'|\leq 5\cdot|P|$ simultaneously computing all $f,\partial_1 f,\ldots,\partial_n f$.*

**Theorem 5.9 (Multipoint Evaluation).** *Let $\mathcal{S}=\big(\mathbb{C},\mathbb{S}^1,(+,\times,\div\big)$.*

a) *Let $\mathbb{F}$ denote a field of characteristic 0 and $\bar u,\bar v\in\mathbb{F}[X]$ such that $\bar u\cdot\bar v\equiv 1 \bmod X^n$. Then $\bar u\cdot(2\bar v-\bar u\cdot\bar v2)\equiv 1 \bmod X^{2n}$.*
b) *There is a straight-line program over $\mathcal{S}$ of length $\mathcal{O}(n\cdot\log n)$ which, given $u_0,u_1,\ldots,u_{n-1}\in\mathbb{C}$ with $u_0\neq 0$, calculates the unique $v_0,\ldots,v_{n-1}\in\mathbb{C}$ such that $(\sum_{k=0}^{n-1}u_kX^k)\cdot(\sum_{k=0}^{n-1}v_kX^k)\equiv 1 \bmod X^n$.*
c) *Let $n\geq m$. There is a straight-line program over $\mathcal{S}$ of length $\mathcal{O}(n\cdot\log n)$ which, given $a_0,\ldots,a_n\in\mathbb{C}$ and $b_0,\ldots,b_m\in\mathbb{C}$ with $b_m\neq 0$, calculates the unique $q_0,\ldots,q_{n-m}\in\mathbb{C}$ and $r_0,\ldots,r_{m-1}\in\mathbb{C}$ such that*

$$\sum_{k=0}^{n}a_kX^k \;=\; \Big(\sum_{k=0}^{m}b_kX^k\Big)\cdot\Big(\sum_{k=0}^{n-m}q_kX^k\Big)+\Big(\sum_{k=0}^{m-1}r_kX^k\Big)$$

d) *There is a straight-line program over $\mathcal{S}$ of length $\mathcal{O}(n\cdot\log^2 n)$ which, given $a_0,\ldots,a_{n-1}\in\mathbb{C}$ and $x_1,\ldots,x_n\in\mathbb{C}$, simultaneously calculates all $\sum_{k=0}^{n-1}a_kx_\ell^k$, $1\leq\ell\leq n$.*

### 5.3 Matrix Multiplication and Tensor Rank

**Example 5.10 (Strassen)** *Let $\mathcal{S}=\big(R,(0,1),(+,\times)\big)$ denote a ring.*

a) *For $A=(A_{ij}),B=(B_{ij})\in R^{2\times 2}$ it holds $A\cdot B=C$ where*

$$C_{11}=M_1+M_4-M_5+M_7,\qquad C_{12}=M_3+M_5,$$
$$C_{21}=M_2+M_4,\qquad C_{22}=M_1-M_2+M_3+M_6$$

$$M_1:=(A_{12}+A_{22})\cdot(B_{11}+B_{22}),\quad M_2:=(A_{21}+A_{22})\cdot B_{11},$$
$$M_3:=A_{11}\cdot(B_{12}-B_{21}),\quad M_4:=A_{22}\cdot(B_{21}-B_{11}),\quad M_5:=(A_{11}+A_{12})\cdot B_{22},$$
$$M_6:=(A_{21}-A_{11})\cdot(B_{11}+B_{12}),\quad M_7:=(A_{12}-A_{22})\cdot(B_{21}+B_{22})$$

b) *Since $R':=R^{n\times n}$ is itself a ring, two matrices over $R^{2n\times 2n}$ can be multiplied using 7 multiplications and a constant number of additions of $n\times n$-matrices.*
c) *$N\times N$ matrix multiplication over $R$ can be performed by a straight-line program over $\mathcal{S}$ of length $\mathcal{O}(n^{\log_2 7})\leq\mathcal{O}(n^{2.81})$.*

**Example 5.11 (Matrix Rank and Tensors)** *Fix a field $\mathbb{F}$ of characteristic 0.*

a) *For finite-dimensional $\mathbb{F}$-vectors spaces $X$ and $Y$ and a linear map $T : X \to Y$, it holds*

$$\mathrm{rank}(T) \quad = \quad \min\Big\{ r \in \mathbb{N} \;\Big|\; \exists \boldsymbol{a}_1, \ldots, \boldsymbol{a}_r \in X \; \exists \boldsymbol{b}_1, \ldots \boldsymbol{b}_r \in Y : \;\; T = \sum\nolimits_{j=1}^{r} \boldsymbol{b}_j \cdot \boldsymbol{a}_j^\dagger \Big\} \quad .$$

b) *Consider $N, M, K \in \mathbb{N}$ and, for $\boldsymbol{a} \in \mathbb{F}^N, \boldsymbol{b} \in \mathbb{F}^M, \boldsymbol{c} \in \mathbb{F}^K$, the $(N \times M \times K)$-hypermatrix $T = (t_{n,m,k})_{1 \leq n \leq N, 1 \leq m \leq M, 1 \leq k \leq K}$ with $t_{n,m,k} := a_n \cdot b_m \cdot c_k$.*

c) *Fix finite-dimensional $\mathbb{F}$-vector spaces $X, Y, Z$ with respective bases $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$, $(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_M)$, and $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_K)$ and algebraic duals $X^*, Y^*, Z^*$. A $(N \times M \times K)$-hypermatrix $T \in \mathbb{F}^{N \times M \times K}$ gives rise to a bilinear map $T : X^* \times Y^* \to Z$ via*

$$X^* \times Y^* \ni (\boldsymbol{x}^*, \boldsymbol{y}^*) \mapsto \sum\nolimits_{n=1}^{N} \sum\nolimits_{m=1}^{M} \sum\nolimits_{k=1}^{K} t_{n,m,k} \cdot \boldsymbol{x}^*[\boldsymbol{x}_n] \cdot \boldsymbol{y}^*[\boldsymbol{y}_m] \cdot \boldsymbol{z}_k \quad .$$

*And, conversely, any bilinear map $T : X^* \times Y^* \to Z$ has a representation (w.r.t. fixed bases) as a $N \times M \times K$-*hypermatrix.

**Definition 5.12 (Tensor Rank).** *Fix a field $\mathbb{F}$ of characteristic 0 and finite-dimensional $F$-vectorspaces $X, Y, Z$ with algebraic duals $X^*, Y^*, Z^*$. A* tensor *is a bilinear map from $X^* \times Y^*$ to $Z$. A* simple *tensor is of the form*

$$\boldsymbol{x} \otimes \boldsymbol{y} \otimes \boldsymbol{z} : X^* \times Y^* \ni (\boldsymbol{u}^*, \boldsymbol{v}^*) \mapsto \boldsymbol{u}^*[\boldsymbol{x}] \cdot \boldsymbol{v}^*[\boldsymbol{y}] \cdot \boldsymbol{z}, \qquad \boldsymbol{x} \in X, \boldsymbol{y} \in Y, \boldsymbol{z} \in Z$$

*and has* rank $\leq 1$; rank $= 0$ *iff $(\boldsymbol{x}, \boldsymbol{y}) = 0$ or $\boldsymbol{z} = 0$. We denote by $X \otimes Y \otimes Z$ the set of tensors $T : X^* \times Y^* \to Z$. The* rank *of such a $T$ is the least $r \in \mathbb{N}$ such that $T$ can be written as the sum of $r$ simple tensors.*

**Lemma 5.13.** a) *Each trilinear functional $\hat{T} : X^* \times Y^* \times Z^* \to \mathbb{F}$ corresponds to a unique tensor $T : X^* \times Y^* \to Z$ and vice versa. (In the sequel we tacitly identify $\hat{T}$ with $T$...)*

b) *$T \in X \otimes Y \otimes Z$ has the same rank as*

$$T' \in Y \otimes Z \otimes X, \quad (\boldsymbol{y}^*, \boldsymbol{z}^*) \mapsto \big( X^* \ni \boldsymbol{x}^* \mapsto \boldsymbol{z}^*[T(\boldsymbol{x}^*, \boldsymbol{y}^*)] \in F \big) \in X \quad .$$

c) *Each $T \in X \otimes Y \otimes Z$ has* $\mathrm{rank}(T) \leq \dim(X) \dim(Y)$ *and* $\mathrm{rank}(T) \geq \dim \mathrm{range}(T)$.

d) *For $X^* := Y^* := Z := \mathbb{F}^{n \times n}$, the* tensor of $n \times n$-matrix multiplication

$$\mathcal{M}_n : X \times Y \to Z, \quad (A, B) \mapsto A \cdot B$$

*has* $\mathrm{rank}(\mathcal{M}_2) \leq 7$ *and* $\mathrm{rank}(\mathcal{M}_{2n}) \leq 7 \cdot \mathrm{rank}(\mathcal{M}_n)$, *hence* $\mathrm{rank}(\mathcal{M}_n) \leq n^{\lceil \log_2 7 \rceil}$.

e) *Let $T \in X \otimes Y \otimes Z$ and $S \in X' \otimes Y' \otimes Z'$. Then*

$$T \oplus S : (X \oplus X') \times (Y \oplus Y') \to Z \oplus Z', \quad \big( (\boldsymbol{x}^*, \boldsymbol{x}'^*), (\boldsymbol{y}^*, \boldsymbol{y}'^*) \big) \mapsto T(\boldsymbol{x}^*, \boldsymbol{y}^*) \oplus S(\boldsymbol{x}'^*, \boldsymbol{y}'^*),$$

$$T \otimes S : (X \otimes X') \times (Y \otimes Y') \to Z \otimes Z', \quad \big( (\boldsymbol{x}^* \otimes \boldsymbol{x}'^*), (\boldsymbol{y}^* \otimes \boldsymbol{y}'^*) \big) \mapsto T(\boldsymbol{x}^*, \boldsymbol{y}^*) \otimes S(\boldsymbol{x}'^*, \boldsymbol{y}'^*)$$

*have* $\mathrm{rank}(T \oplus S) \leq \mathrm{rank}(T) + \mathrm{rank}(S)$ *and* $\mathrm{rank}(T \otimes S) \leq \mathrm{rank}(T) \cdot \mathrm{rank}(S)$.

**Theorem 5.14 (Exponent of Matrix Multiplication).** *Fix a field $F$ of characteristic 0 and $\omega \geq 2$ as well as the structure $\mathcal{S} = \big(F, F, (+, \times)\big)$. The following are equivalent:*

i) *To every $\epsilon > 0$ there exists a family $P_n$ of straight-line programs over $\mathcal{S}$ of length $\mathcal{O}(n^{\omega+\epsilon})$ which, given $A, B \in F^{n \times n}$, calculate $A \cdot B$.*

ii) *To every $\epsilon > 0$, it holds* $\mathrm{rank}(\mathcal{M}_n) \leq \mathcal{O}(n^{\omega+\epsilon})$.

# 6 Branching Complexity

**Definition 6.1.** *Let* $\mathcal{S} = \big(S, (c_i), (f_j), (P_k)\big)$ *denote a structure with relations* $P_k :\subseteq S^{b_k}$ *or arities* $b_k \in \mathbb{N}$.

a) *A* Branching Tree $T_{\mathcal{S}}$ *(over this structure and in variables* $X_1, \ldots, X_n$*) is basically a straight-line program with the additional capability to branch based on whether a predicate* $P_k$*, applied to previously calculated results, holds or not.*
*More formally, it is a rooted binary tree whose outdegree-1 nodes* $u \in T_{\mathcal{S}}$ *are each labelled with either a variable, a constant* $c_i$ *from* $\mathcal{S}$*, or with a function* $f_j$ *applied to results from* $a_j$ *outdegree-1 predecessor nodes of* $u$*; and each outdegree-2 node is labelled with a predicate* $P_k$ *applied to* $b_k$ *degree-1 predecessor nodes. Each leaf (=outdegree-0 node) is labelled either with some symbol* $\sigma \in \Sigma$ *or with some finite tuple of degree-1 predecessor nodes.*
b) *When assigned values* $x_1, \ldots, x_n \in S$ *to* $X_1, \ldots, X_n$ *the tree calculates, starting from the root, in outdegree-1 nodes intermediate results; and in outdegree-2 nodes branches according to whether the predicate holds.* $T_{\mathcal{S}}$ accepts *input* $\boldsymbol{x} \in S^n$ *if this process ends in a leaf labelled* + $\in \Sigma$*; it* rejects *if the leaf is labelled* − $\in \Sigma$*; otherwise it* computes *the specified (tuple of intermediate) value(s).*
c) *The* size *of a branching tree is its total number of nodes; similarly for the* depth.

**Example 6.2 (Sorting)** *Consider some totally ordered set* $S$ *and the structure* $\mathcal{S} = \big(S, (), (), (<)\big)$*. We say that a branching tree over* $\mathcal{S}$ *on* $n$ *variables* sorts *if it computes some function* $(f_1, \ldots, f_n) = \bar{f} : S^n \to S^n$ *such that, for every* $\bar{x} = (x_1, \ldots, x_n) \in S^n$,

$$f_1(\bar{x}) \leq f_2(\bar{x}) \leq \ldots \leq f_n(\bar{x}) \;\; \text{and} \;\; \forall y \in S: \; \#\{j : x_j = y\} = \#\{j : f_j(x_1, \ldots, x_n) = y\} \tag{1}$$

a) *For each* $n \in \mathbb{N}$*, both* Bubble Sort *and* Quicksort *give rise to branching trees over* $\mathcal{S}$ *in* $n$ *variables of depth* $\mathcal{O}(n^2)$.
b) Heap Sort *gives rise to a branching tree over* $\mathcal{S}$ *in* $n$ *variables of depth* $\mathcal{O}(n \cdot \log n)$.
c) *If* $|S| \geq n$*, then any branching tree over* $\mathcal{S}$ *in* $n$ *variables has at least* $n!$ *different leaves. In particular,* Heap Sort *is asymptotically optimal.*

## 6.1 Hyperplane Arrangements and Combinatorial Convex Geometry

**Definition 6.3.** *Fix* $d \in \mathbb{N}$.

a) *A set* $X \subseteq \mathbb{R}^d$ *is* convex *if*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in X : \; \lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in X \tag{2}$$

*holds for every* $0 \leq \lambda \leq 1$. $X$ *is* affine *if Equation (2) holds for every* $\lambda \in \mathbb{R}$*; equivalently:* $X \neq \emptyset$ *and* $X - \boldsymbol{y} := \{\boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x} \in X\}$ *is a vector space for some/every* $\boldsymbol{y} \in X$.
b) *We call* $\boldsymbol{h} \in \mathbb{S}^d = \{\boldsymbol{y} \in \mathbb{R}^{d+1} : \|\boldsymbol{y}\| = 1\}$ *an* oriented hyperplane*. Its* open halfspace $H_{<\boldsymbol{h}}$ *is the set* $\big\{\boldsymbol{x} \in \mathbb{R}^d : \sum_j x_j \cdot h_j < h_0\big\}$*; the topological closure* $H_{\leq \boldsymbol{h}} := \overline{H_{<\boldsymbol{h}}}$ *its* closed halfspace*. Finally write* $H_{=\boldsymbol{h}} := H_{\leq \boldsymbol{h}} \cap H_{\leq -\boldsymbol{h}}$ *for its affine hyperplane.*

c) *The* dimension *of $X \subseteq \mathbb{R}^d$, $\dim(X)$, is the affine dimension of* $\mathrm{ahull}(X) := \{\lambda \cdot \boldsymbol{x} + (1-\lambda)\cdot \boldsymbol{y} : \boldsymbol{x}, \boldsymbol{y} \in X, \lambda \in \mathbb{R}\}$*;* $\dim(\emptyset) := -\infty$. *A (convex)* polytope *$P \subseteq \mathbb{R}^d$ is the finite intersection of finitely many open/closed halfspaces.*

d) *The* membership problem *associated with a finite family $\mathcal{H}$ of affine hyperplanes in $\mathbb{R}^d$ is the question of whether a given $\boldsymbol{x} \in \mathbb{R}^d$ belongs to $\bigcup \mathcal{H}$ or not.*

e) *For $\boldsymbol{h} \in \mathbb{S}^d$ and $\boldsymbol{x} \in \mathbb{R}^d$, write* $\mathrm{sgn}(x, \boldsymbol{h}) := \mathrm{sgn}(\sum_j x_j \cdot h_j - h_0) \in \{+, 0, -\}$.

f) *The* point location problem *associated with a finite family $\mathcal{H} = \{\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(n)}\}$ of oriented hyperplanes is the function*

$$\mathbb{R}^d \ni \boldsymbol{x} \mapsto \mathrm{sgn}(\boldsymbol{x}, \mathcal{H}) := \big(\mathrm{sgn}(x, \boldsymbol{h}^{(k)})\big)_{1 \le k \le n} \in \{+, 0, -\}^n .$$

g) *A* face *of $\mathcal{H}$ is a subset of $\mathbb{R}^d$ of the form*

$$\mathcal{H}(\bar{\sigma}) := \big\{\boldsymbol{x} \in \mathbb{R}^d : \mathrm{sgn}(\boldsymbol{x}, \mathcal{H}) = \bar{\sigma}\big\}, \qquad \bar{\sigma} \in \{+, 0, -\}^n .$$

*A face of dimension 0 is called a* vertex*; an* edge *is a face of dimension 1; a face of dimension $d$ is a* cell*; a* facet *is a face of dimension $d-1$; a face of dimension $d-2$ is called* ridge.

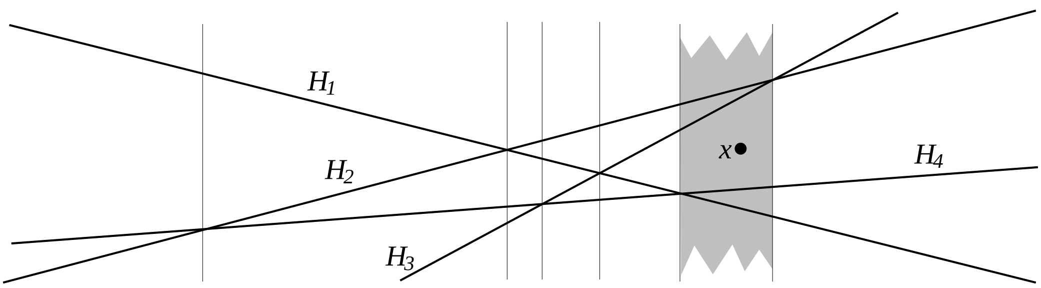


**Fig. 3.** An arrangement of 4 lines in the plane inducing 6 intersections (=vertices), 16 line segments (=edges=ridges=facets), and 11 cells.

**Lemma 6.4.** *Fix a finite family $\mathcal{H}$ of $n$ oriented hyperplanes in $\mathbb{R}^d$.*

a) *Each face of $\mathcal{H}$ is a polytope.*

b) *Each vertex of $\mathcal{H}$ is determined by (at least) $d$ hyperplanes.*
*In particular, $\mathcal{H}$ has at most $\binom{n}{d}$ vertices.*

c) *For an arrangement of $n$ hyperplanes in dimension $d$, the number of $k$-dimensional faces is at most $\sum_{j=0}^{k} \binom{d-j}{k-j} \cdot \binom{n}{d-j}$*

d) *and these numbers are attained by almost every arrangement.*

**Example 6.5** *For $2 < N \in \mathbb{N}$, the following 2D arrangement has a cell with $N$ facets:*

$$\boldsymbol{h}_n := (1, \cos\tfrac{2\pi n}{N}, \sin\tfrac{2\pi n}{N})/\sqrt{2}, \quad 0 \le n < N .$$

### 6.2 Linear Branching Trees

**Definition 6.6.** *A Linear Branching Tree for dimension $d \in \mathbb{N}$ is a branching tree over the structure $\mathcal{S} := \big(\mathbb{R}^d, (), (), (H_{=\boldsymbol{h}}, H_{<\boldsymbol{h}} : \boldsymbol{h} \in \mathbb{S}^d)\big)$.*

**Example 6.7** *To each $n$-element family $\mathcal{H}$ of oriented hyperplanes in $\mathbb{R}^d$, there exists a Linear Branching Tree of depth $\mathcal{O}(n)$ deciding the membership problem associated with $\mathcal{H}$.*

**Lemma 6.8.** *Let $T$ denote a linear branching tree for dimension $d$ and $v$ a vertex of $T$. Write $T(v)$ for the set of inputs $\boldsymbol{x} \in \mathbb{R}^d$ which, according to the semantics of Definition 6.1b), passes through $v$.*

*a) $T(v)$ is a polytope. Each facet corresponds to an oriented hyperplanes queried by $T$ on the path from the root up to $v$.*
*b) For the leaves $v_1, \ldots, v_N$ of $T$, $\big(T(v_j)\big)_{j=1,\ldots,N}$ constitutes a partition of $\mathbb{R}^d$.*
*c) For any linear branching tree $T$ over $\mathcal{S}$ solving membership to $\mathcal{H}$, and for each leaf $v$ of $T$, $T(v)$ is either a subset of some $H_{=\boldsymbol{h}}$ with $\boldsymbol{h} \in H$ or of $\mathcal{H}(\bar{\sigma})$ for some $\bar{\sigma} \in \{+,-\}^{\mathcal{H}}$.*

**Theorem 6.9 (Ukkonen'83, Dobkin/Lipton'74, Meiser'93).**
*Fix an $n$-element family $\mathcal{H}$ of oriented hyperplanes in dimension $d$.*

*a) Suppose $\mathcal{H}$ has $N$ distinct cells. Then any linear branching tree over $\mathcal{S}$ deciding membership to $\mathcal{H}$ has depth at least $\log N$.*
*b) Let $\mathcal{H}(\bar{\sigma})$ denote a cell having $m$ facets. Then any linear branching tree over $\big(\mathbb{R}^d, (), (), (H_{=\boldsymbol{h}}, H_{<\boldsymbol{h}} : \boldsymbol{h} \in \mathcal{H})\big)$ deciding membership to $\mathcal{H}$ has depth at least $m$.*
*c) There exists a linear branching tree over $\mathcal{S}$ of depth $\mathcal{O}(\log n)$ solving the point location problem for $\mathcal{H}$.*
*d) There exists a linear branching tree over $\mathcal{S}$ of depth $\mathcal{O}(d^5 \log n)$ solving the point location problem for $\mathcal{H}$.*

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