Martin Ziegler Issued on 2011-05-27
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Advanced Complexity Theorie

SS 2011, Exercise Sheet #7

EXERCISE 10:

a) Recall the n-dimensional discrete Fourier-transform

$$\mathcal{F}_n : \mathbb{C}^n \ni (x_0, \ldots, x_{n-1}) \mapsto \left(\sum_{\ell=0}^{n-1} \exp(2\pi i \cdot k \cdot \ell/n) \cdot x_\ell\right)_{k=0,\ldots,n-1} \in \mathbb{C}^n$$

Show that $\mathcal{F}_n\big(\mathcal{F}_n^*(\vec{x})\big) = n \cdot \vec{x}$ where $\mathcal{F}_n^*$ denotes its transpose. Conclude $|\det(\mathcal{F}_n)| = n^{n/2}$.

b) Look up the (Cooley-Tukey) *Fast Fourier Transform* algorithm and verify that it yields a straight-line program for $\mathcal{F}_n$ over $\mathcal{S} := \big(\mathbb{C}, (), (+, \times_c : c \in \mathbb{S}^1)\big)$ of length $\mathcal{O}(n \cdot \log n)$.

EXERCISE 11:

a) Let F denote a field of characteristic 0, $X \subseteq F$ of cardinality $K \in \mathbb{N}$, and $p := \sum_{0 \leq k_1,\ldots,k_n < K} a_{k_1,\ldots,k_n} X_1^{k_1} \cdots X_n^{k_n} \in F[X_1, \ldots, X_n]$ a multivariate polynomial with $p(\vec{x}) = 0$ for every $x_1, \ldots, x_n \in X$. Show that this requires $a_{\vec{k}} = 0$ for every $\vec{k}$.

b) Let $X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{m-1}$ denote variables. Show that the $2n + 2m$ elements

$$1,\ X_0, \ldots, X_{n-1},\ Y_0, \ldots, Y_{m-1},\ \sum_{k=0}^{\ell} X_k \cdot Y_{\ell-k} \qquad (\ell = 0, \ldots, n+m-2)$$

of $F[X_1, \ldots, Y_m]$ are linearly independent over F.

Fix fields $F \subseteq E$ and recall that $x_1, \ldots, x_n \in E$ are called *algebraically dependent* (over F) if there exists a non-zero n-variate polynomial $p \in F[X_1, \ldots, X_n]$ such that $p(x_1, \ldots, x_n) = 0$. A set is algebraically *in*dependent over F if no finite subset is algebraically dependent over F. The *transcendence degree* over F of a set $S \subseteq E$, $\text{trdeg}_F(S)$, denotes the cardinality of a largest subset of S algebraically independent over F.

EXERCISE 12:

Recall that S is algebraic over F iff every $e \in S$ is the root of some $0 \neq p \in F[X]$;
equivalently: $F(e)$ has finite dimension as an F-vectorspace.

a) Show that $\text{trdeg}_F(S)$ coincides with the least cardinality of a subset $T \subseteq S$ such that S is algebraic over $F(T)$. (You may restrict to the case $\text{trdeg}_F(S) < \infty\ldots$)

b) Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in E$ such that each y_j is algebraic over $F(x_1, \ldots, x_n)$ and $z \in E$ is algebraic over $F(y_1, \ldots, y_m)$. Show that z is algebraic over $F(x_1, \ldots, x_n)$.

c) For a further field $G \subseteq F$, conclude $\text{trdeg}_G(E) = \text{trdeg}_F(E) + \text{trdeg}_G(F)$.

d) Determine the transcendence degree of $\mathbb{C}$ over $\mathbb{R}$ and that of $\mathbb{R}$ over $\mathbb{Q}$.

e) Suppose F has characteristic 0 and consider the set $E := F(X_1, \ldots, X_n)$ of n-variate rational functions. Determine its transcendence degree over F. What happens in finite characteristic?

f) Is $\{\pi, e\}$ algebraically dependent over $\mathbb{Q}$ or not?