Prof. H. Alber
Dr. N. Kraynyukova, N. Sissouno

Mathematik III für ETiT, WI(ET), IST, CE, LaB-ET, Sport-Wiss

2.Übung

Präsenzaufgaben

G1 (Potentiale)

(i) $F(x,y) = (2x, 2y)^T$

$\frac{d}{dy}(2x) = \frac{d}{dx}(2y)$ ✓ Potential!

$\nabla \varphi = F$, $\varphi_x = 2x$, $\varphi_y = 2y$

$\Rightarrow \varphi(x,y) = 2x^2 + \psi(y)$

$\Rightarrow \varphi_y(x,y) = \psi_y(y) \overset{!}{=} 2y$, d.h. $\psi(y) = 2y^2 + C$

$\rightarrow \underline{\underline{\varphi(x,y) = x^2 + y^2 + C}}$

(ii) $F(x,y) = (2y, 2x)^T$

$\frac{d}{dy}(2y) \overset{!}{=} \frac{d}{dx}(2x)$ ✓ Potential!

$\varphi_x = 2y \Rightarrow \varphi(x,y) = 2xy + \psi(y)$

$\varphi_y(x,y) = 2x + \psi' \overset{!}{=} 2x$, d.h. $\psi'(y) = 0$

$\rightarrow \underline{\underline{\varphi(x,y) = 2xy + C}}$

(iii) $F(x,y) = (x, xy)$

$\frac{d}{dy}(x) \overset{!}{=} \frac{d}{dx}(xy)$ $\lightning$ Kein Potentialfeld

(iv) $F(x,y,z) = (z\cos y, -zx\sin y + z, x\cos y + y)^T$

$\frac{d}{dy}(z\cos y) \overset{!}{=} \frac{d}{dx}(-zx\sin y + z)$ ✓

$\frac{d}{dz}(z\cos y) \overset{!}{=} \frac{d}{dx}(x\cos y + y)$ ✓ Potential!

$\frac{d}{dz}(-zx\sin y + z) \overset{!}{=} \frac{d}{dy}(x\cos y + y)$ ✓

$\varphi_x \overset{!}{=} z\cos y \Rightarrow \varphi(x,y,z) = xz\cos y + \psi(y,z)$

$\rightsquigarrow \varphi_y = -xz\sin y + \psi_y \overset{!}{=} -zx\sin y + z$

also $\psi_y = z$, $\psi = yz + \chi(z)$, d.h.

$\varphi(x,y,z) = xz\cos y + yz + \chi(z)$

$\rightsquigarrow \varphi_z = x\cos y + y + \chi' \overset{!}{=} x\cos y + y$

also $\chi' = 0$, $\chi = C$

insgesamt: $\varphi(x,y,z) = \underline{\underline{xz\cos y + yz + C}}$

G2 (Potential)

i) $\frac{\partial (F_a)_2}{\partial x} \overset{!}{=} \frac{\partial (F_a)_1}{\partial y}$

$\frac{\partial (F_a)_2}{\partial x} = e^{x+y} + 2x$

$\frac{\partial (F_a)_1}{\partial y} = e^{x+y} + ax$

$\Rightarrow a = 2$

$\int (e^{x+y} + 2xy)\, dx = e^{x+y} + x^2 y + g(y)$

Ableiten nach y:

$\frac{\partial}{\partial y}(e^{x+y} + x^2 y + g(y)) = e^{x+y} + x^2 + g_y(y) \overset{!}{=} e^{x+y} + x^2$

$\Rightarrow g_y(y) = 0 \quad \Rightarrow g(y) = \text{const.} = c$

$\Rightarrow$ Potential $\varphi(x,y) = e^{x+y} + x^2 + c$.

ii) Direkte Integration wäre sehr unangenehm.

Stattdessen schreiben wir:

$F_0 = F_2 + \begin{pmatrix} -2xy \\ 0 \end{pmatrix}$ und F_2 ist ja ein Potential.

Also

$W = \int_K F_0(x) \cdot dx = \int_K F_2(x)\, dx - \int_K \begin{pmatrix} 2xy \\ 0 \end{pmatrix} dx$

$= \varphi(1,1) - \varphi(0,0) - \int_0^1 \begin{pmatrix} 2t^5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} dt =$

$= e^2 + 1 - e^0 - \int_0^1 4t^6\, dt = e^2 - 4\frac{t^7}{7}\Big|_0^1$

$= e^2 - \frac{4}{7}$

Hausaufgaben

H1 (Punktladung)

i) $\frac{\partial E_y}{\partial x} = \frac{q}{4\pi\epsilon_0}\frac{xy}{||X||^5} = \frac{\partial E_x}{\partial y}, \quad \frac{\partial E_x}{\partial z} = \frac{q}{4\pi\epsilon_0}\frac{xz}{||X||^5} = \frac{\partial E_z}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{q}{4\pi\epsilon_0}\frac{yz}{||X||^5} = \frac{\partial E_y}{\partial z}$

$\Rightarrow$ Es gibt ein Potential!

ii) $\frac{\partial E_i}{\partial x_j} = h'(||X||)\frac{x_i x_j}{||X||}$

iii) $\psi(X) = \frac{q}{4\pi\epsilon_0}\frac{1}{||x||}$

iv) Wir verwenden Kugelkoordinaten $\rightsquigarrow \quad \psi(r,\varphi,\theta) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$.

$$-\int_{r_1}^{r_2} q' F(r,\varphi,\theta)\, dr = -\frac{qq'}{4\pi\epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

v) $\lim_{R\to\infty} -\int_1^R \frac{qq'}{4\pi\epsilon_0}\frac{1}{r}\, dr = \frac{qq'}{4\pi\epsilon_0}1$.

H2 (Vektoranalysis)

Sei $\alpha(u,v)$ der Winkel zwischen $\varphi_u = (x'_u, y'_u, z'_u)^T$ und $\varphi_v = (x'_v, y'_v, z'_v)^T$. Dann ist

i)

$$\begin{aligned} \|\varphi_u \times \varphi_v\| &= \|\varphi_u\| \cdot \|\varphi_v\| \cdot \sin\alpha, \quad 0 < \alpha < \pi \\ \sqrt{\|\varphi_u\|^2\|\varphi_v\|^2 - \langle\varphi_u,\varphi_v\rangle^2} &= \sqrt{\|\varphi_u\|^2\|\varphi_v\|^2 - \|\varphi_u\|^2\|\varphi_v\|^2\cos^2\alpha} \\ &= \|\varphi_u\| \cdot \|\varphi_v\|\sqrt{\sin^2\alpha} = \|\varphi_u\| \cdot \|\varphi_v\| \cdot \sin\alpha. \end{aligned}$$

ii)

$$\varphi_u = (1, 0, z'_u)^T,\ \varphi_v = (0, 1, z'_v)^T$$

$$\varphi_u \times \varphi_v = \begin{pmatrix} i & j & k \\ 1 & 0 & z'_u \\ 0 & 1 & z'_v \end{pmatrix} = i(-z'_u) + j(-z'_v) + k \cdot 1 = \begin{pmatrix} -z'_u \\ -z'_v \\ 1 \end{pmatrix}$$

Also ist $\|\varphi_u \times \varphi_v\| = \sqrt{(z'_u)^2 + (z'_v)^2 + 1}$.

H3 (Oberfläche)

$$\varphi_u = \begin{pmatrix} u^2 - 1 \\ 2u \\ 0 \end{pmatrix}, \quad \varphi_v = \begin{pmatrix} 0 \\ 0 \\ 3v^2 \end{pmatrix}.$$

Nach dem Ergebnis H2 a) gilt:

$$\begin{aligned} \|\varphi_u \times \varphi_v\| &= \sqrt{\|\varphi_u\|^2 \cdot \|\varphi_v\|^2 - \langle \varphi_u, \varphi_v \rangle^2} \\ &= \sqrt{((u^2-1)^2 + 4u^2) \cdot 9v^4} = 3v^2\sqrt{(u^2+1)^2} = 3v^2(u^2+1). \end{aligned}$$

Dann berechnet man den Oberflächeninhalt $I(S)$ der durch die $\varphi(u, v)$ gegebenen Oberfläche S:

$$\begin{aligned} I(S) &= \iint_S d\sigma = \iint_D 3v^2(u^2+1) d(u, v) \\ &= \int_0^1 dv \int_0^v 3v^2(u^2+1) du = 3\int_0^1 v^2 dv \Big(\frac{v^3}{3} + v\Big) \\ &= \frac{v^6}{6}\Big|_0^1 + 3\frac{v^4}{4}\Big|_0^1 = \frac{1}{6} + \frac{3}{4} = \frac{11}{12}. \end{aligned}$$