24.4.1

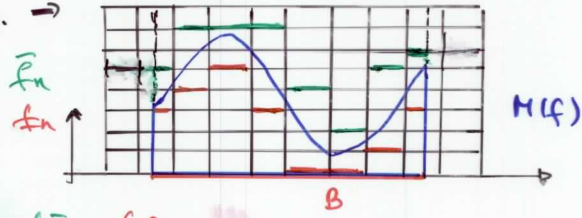
$B \subseteq \mathbb{R}^n$ messbar, $f : B \to \mathbb{R}$

Ordinatenmenge $M(f) = \{(x,y) \mid 0 \le y \le f(x), x \in B\}$

$M(f)$ messbar $\Leftrightarrow$ f integrierbar

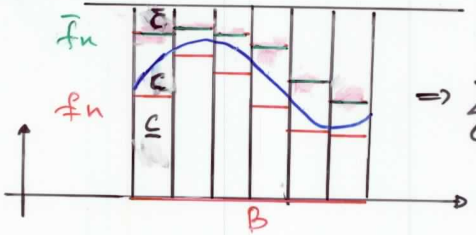
$\mu(M(f)) = \int_B f$

Bew. $\Rightarrow$



$\Rightarrow \int_B \bar{f}_n - \int_B \underline{f}_n \to 0$

$\Leftarrow$

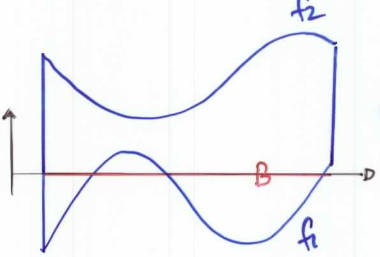


$\Rightarrow \sum_C \mu(C) \to 0$

$\Rightarrow$ Graph f Nullmenge

$f_1 \leq f_2$

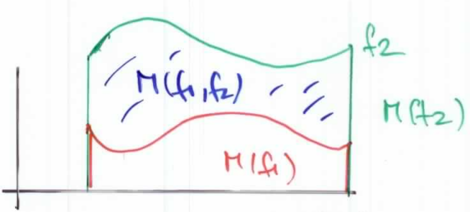
$M(f_1, f_2) = \{(x,y) \mid f_1(x) \leq y \leq f_2(x), x \in B\}$



f_1, f_2 integrierbar $\Rightarrow M(f_1, f_2)$ messbar

$$\mu(M(f_1, f_2)) = \int_B f_2 - f_1$$

Bew. oBdA $f_1 \geq 0$



$M(f_1, f_2) = M(f_2) \setminus M(f_1)$

24.4.2

$h_1, h_2 : B \to \mathbb{R}$ stetig, $h_1 \le h_2$

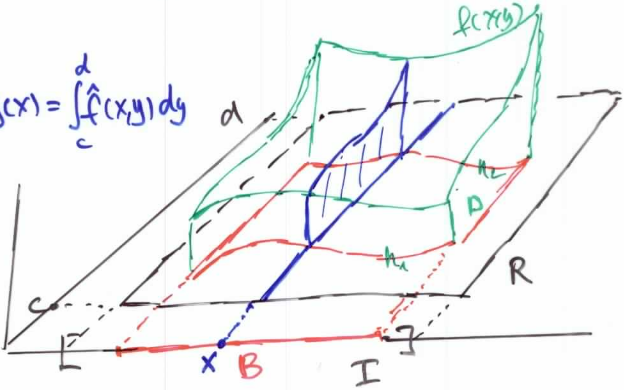
$D = M(h_1, h_2)$ relativer Normalbereich über B

Satz $f : D \to \mathbb{R}$ integrierbar

$$g(x) = \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy \quad \text{ex. } \forall x \in B$$

$$\to \int_D f(x,y)\,d(x,y) = \int_B g(x)\,dx$$

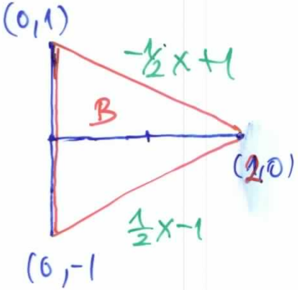
$$\hat{g}(x) = \int_c^d \hat{f}(x,y)\,dy$$



$$\int_D f = \int_R \hat{f} = \int_I \left(\int_c^d \hat{f}(x,y)\,dy \right) dx = \int_I \hat{g}(x)\,dx$$

$$= \int_B g(x)\,dx$$

↑ Fubini



$$f(x,y) = 3xy^2$$

$$\int_0^2 \left(\int_{\frac{1}{2}x-1}^{-\frac{1}{2}x+1} 3xy^2 \, dy \right) dx$$

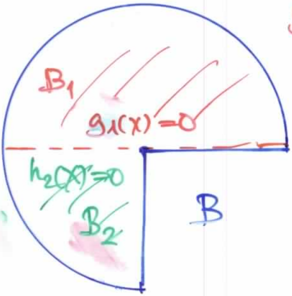
$$= \int_0^2 \left[xy^3 \right]_{\frac{1}{2}x-1}^{-\frac{1}{2}x+1} dx$$

$$= \int_0^2 x\left(-\frac{1}{8}x^3+\frac{3}{4}x^2-\frac{3}{2}x+1\right) - x\left(\frac{1}{8}x^3-\frac{3}{4}x^2+\frac{3}{2}x-1\right)$$

$$= \int_0^2 -\frac{1}{4}x^4+\frac{3}{2}x^3-3x^2+2x \, dx$$

$$= \left[-\frac{1}{20}x^5+\frac{3}{8}x^4-x^3+x^2 \right]_0^2$$

$$= -\frac{8}{5}+6-8+4 = 2-\frac{8}{5} = \frac{2}{5}$$



$g_2(x)=\sqrt{1-x^2}$ 24

$f(x,y)=xy$

$h_1(x)=-\sqrt{1-x^2}$

$$\int_{B_1} f = \int_{-1}^{+1}\int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$

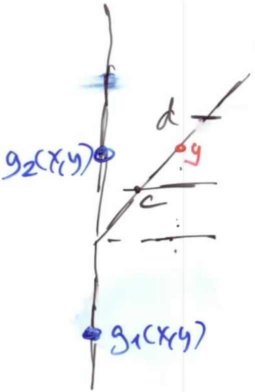
$$= \int_{-1}^{1}\left[\frac{1}{2}xy^2\right]_0^{\sqrt{1-x^2}} dx$$

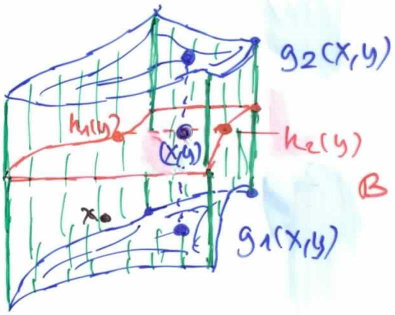
$$= \int_{-1}^{1}\frac{1}{2}x(1-x^2)\,dx = \left[\frac{1}{4}x^2-\frac{1}{8}x^4\right]_1^{-1} = 0$$

$$\int_{B_2} f = \int_{-1}^{0}\int_{-\sqrt{1+x^2}}^{0} xy\,dy\,dx$$

$$= -\left[\frac{1}{4}x^2-\frac{1}{8}x^4\right]_{-1}^{0} = \frac{1}{8}$$

$$\int_B f = \int_{B_1} f + \int_{B_2} f = \frac{1}{8}$$


d
y
c
$g_2(x,y)$
$g_1(x,y)$


$g_2(x,y)$
$h_1(y)$
(x,y)
$h_2(y)$
B
x
$g_1(x,y)$

$[c,d]$

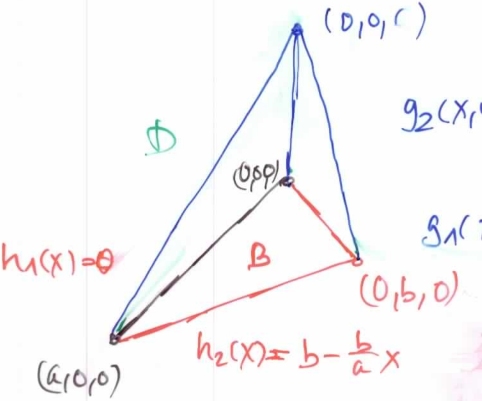
$$B = \{(x,y) \mid y \in [c,d],\ h_1(y) \le x \le h_2(y)\}$$

$$D = \{(x,y,z) \mid (x,y) \in B\ \ g_1(x,y) \le z \le g_2(x,y)\}$$

$$\int_D f(x,y,z)\, d(x,y,z)$$

$$= \int_B \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right) d(x,y)$$

$$= \int_c^d \underbrace{\left(\int_{h_1(y)}^{h_2(y)} \underbrace{\left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right)}_{G(x,y)} dx \right)}_{F(y)} dy$$



$$g_2(x,y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$$g_1(x,y) = 0$$

$$\int_D f(x,y,z)\, d(x,y,z)$$

$$= \int_0^a \left(\int_0^{b-\frac{b}{a}x} \left(\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} f(x,y,z)\, dz \right) dy \right) dx$$

$f = 1$

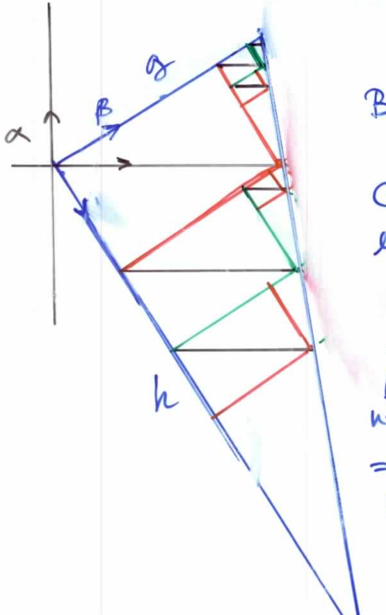
$$= \int_0^a \int_0^{b-\frac{b}{a}x} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$= \int_0^a c\left[y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b-\frac{b}{a}x} = \int_0^a \frac{bc}{2}\left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{abc}{6}$$

(4) D rechtwinklig $\Rightarrow \mu_\alpha(D) = \frac{1}{2} g h$

$A \subseteq B$ A, B messb.
$\Rightarrow \mu(A) \le \mu(B)$



Bew $\boxed{C}$

C Zelle für α

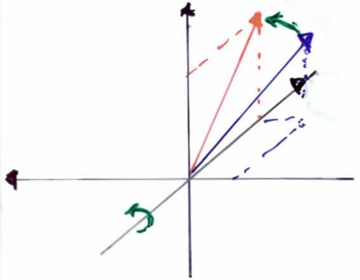
$\mu_\alpha(C) = \mu_\beta(C)$

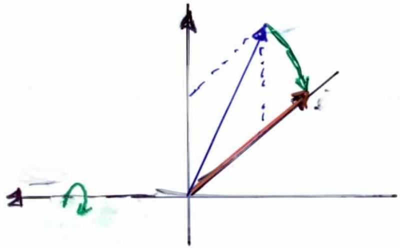
$$\mu_\alpha(D) = \lim_{n\to\infty} \sum_{C \in Z_n} \mu_\alpha(C) = \lim_{n\to\infty} \sum_{C \in Z_n} \mu_\beta(C) = \mu_\beta(D) = \frac{1}{2} g h$$

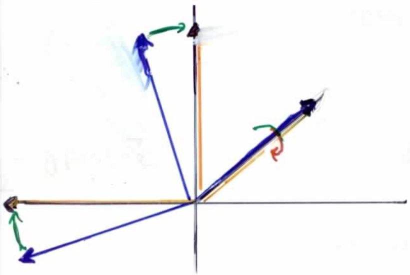
(5) R Rechteck
$\Rightarrow \mu_\alpha(R) = g h$

(6) $\mu_\alpha = \mu_\beta$

$n=3$ Euler





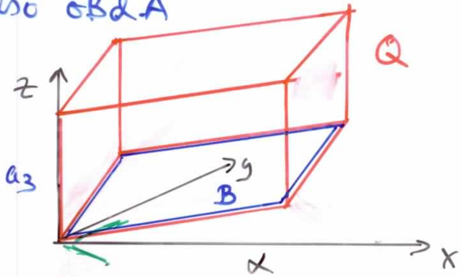


Euler α, β ONB

$\Rightarrow \exists\ \alpha = \beta_0, \dots, \beta_4 = \beta$ ONB

β_i, β_{i+1} haben gemeinsame Achse

also oBdA



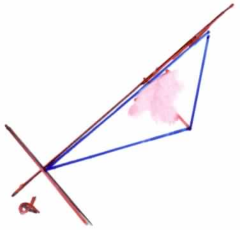
$$\mu_\alpha(Q) = \int_B a_3 \, d(x,y) = a_3 \, \mu(B)$$
$$= a_3 a_1 a_2$$
$$= \mu_\beta(Q)$$

β achsenparallel Q

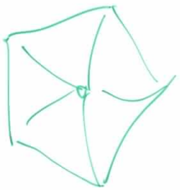
$\Rightarrow \mu_\alpha = \mu_\beta$

25.1.5 Konvexe Polyeder sind messbar

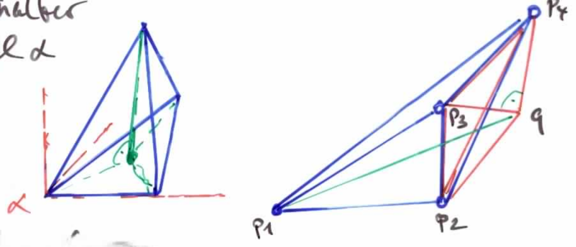
$n=2$



Normalber. bzgl α



Normalber bzgl α



$$P = [p_1, p_2, p_3, p_4, q]$$
$$= [p_1, p_2, p_4, q] \cup [p_1, p_3, p_4, q]$$

$$[p_1, p_2, p_3, p_4] = P \setminus [p_2, p_3, p_4, q]$$

25.3.2

$S = \text{Spat}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$

$\mu(S) = |\det A|$

Bew: $S \rightsquigarrow S'$ Vertauschung $\to S = S'$

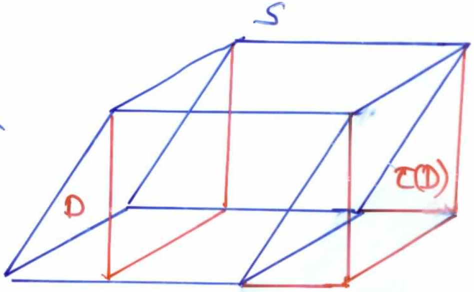
$S \rightsquigarrow S'$ Scherung

$A \rightsquigarrow A'$ $\det A = \det A'$

$\mu(S) = \mu(S')$

τ Translation

D messbar



$S' = (S \setminus D) \cup \tau(D)$

$\mu(S') = (\mu(S) - \mu(D)) + \mu(D)$

Nun $A \rightsquigarrow A''$ Diagonal $|\det A''| = |\det A|$

$S \rightsquigarrow S''$ Quader $\quad \overset{\shortparallel}{\mu(S'')} = \mu(S)$

(3)

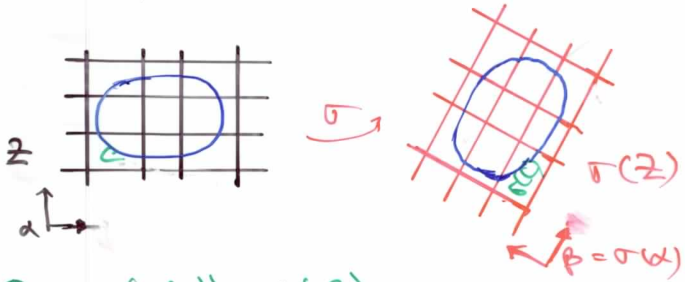
25.3.1

$|\sigma(X)| = |X|$

σ Bewegung, B messbar

$\Rightarrow \sigma(B)$ messbar, $\mu(\sigma(B)) = \mu(B)$

$\sigma(Z) = \{\sigma(C) \mid C \in Z\}$



Bew $\mu(\sigma(C)) = \mu(C)$

$$\Rightarrow \mu_\beta(\sigma(B)) = \mu_\alpha(B) = \mu(B)$$

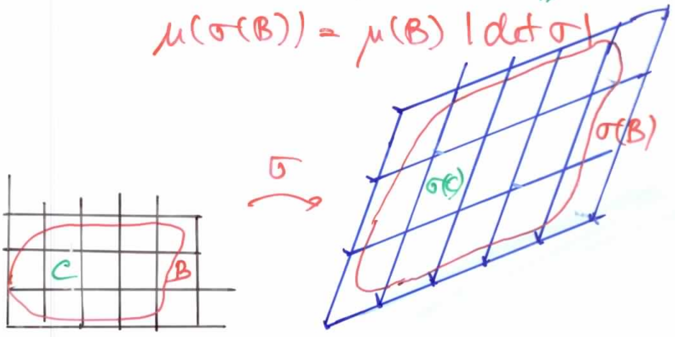
$$\mu_\beta(\sigma(B)) = \mu(\sigma(B))$$

25.3.3

$\sigma(x)^\alpha = A\vec{x} + b$

Satz $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ affin $\Rightarrow$

$$\mu(\sigma(B)) = \mu(B)\,\underbrace{|\det \sigma|}_{=\det A}$$



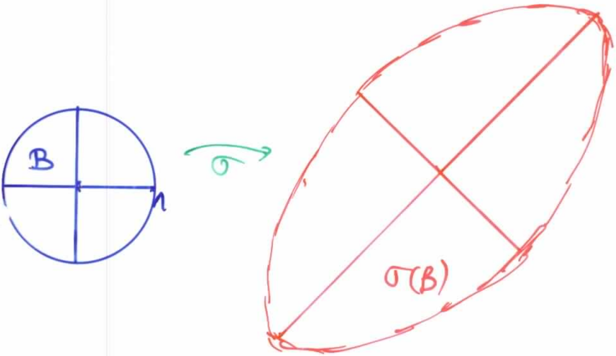
Bew C Quader, $\sigma(C)$ Spat

$$\mu(\sigma(C)) = \mu(C)\,|\det \sigma|$$

Z_n Gitter-Zerlegung von B

$\sigma(Z_n) = \{\sigma(C) \mid C \in Z_n\}$

Zerlegung von $\sigma(B)$

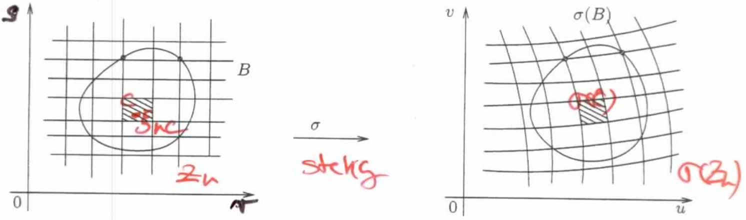


$$\sigma\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}3 & -3\\ 1/2 & 1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\det\sigma = \frac{9}{\cancel{8}\sqrt{2}\cdot 4}$$

$$\mu(\sigma(B)) = \frac{9\,\cancel{\pi^2}}{\cancel{8}\,\sqrt{2}\,4}\,\pi^2$$

23.4 Substitution



$\tau: B \to \mathbb{R}$ stetig

$$\mu(\sigma(C)) = \tau(\xi_{nC})\,\mu(C)$$

$\boxed{\sigma, \tau \text{ Substitution}}$

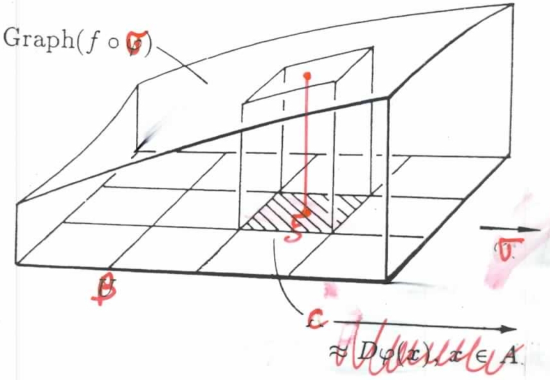
Beisp. σ affin, $\tau(\vec{r}) = \underset{\neq 0}{\det \sigma}$

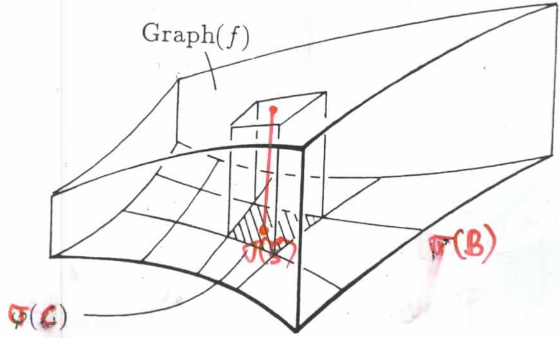
$$\int_{\sigma(B)} f(\vec{u})\, d\vec{u} = \int_B f(\sigma(\vec{r}))\cdot\tau(\vec{r})\, d\vec{r}$$

$$\| \qquad\qquad = \lim_{n\to\infty} \sum_{C\in Z_n} f(\sigma(\xi_{nC}))\,\tau(\xi_{nC})\,\mu(C)$$

$$\lim_{n\to\infty} \sum_{D\in\sigma(Z_n)} f(\xi'_{nD})\,\mu(D)$$

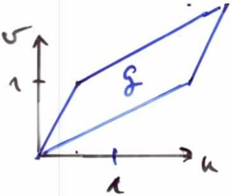
mit $\xi'_{nD} = \sigma(\xi_{nC})$





$\mu(\sigma(C)) = \tau(\xi)\mu(C)$

15



$$f\begin{pmatrix} u \\ v \end{pmatrix} = u + v^2$$

$$G = \sigma(B) \qquad B = [0,1] \times [0,1]$$

$$\sigma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1/2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\tau\begin{pmatrix} x \\ y \end{pmatrix} = \det\begin{pmatrix} 2 & 1/2 \\ 1 & 1 \end{pmatrix} = \frac{3}{2}$$

$$\int_G f(u,v)\, d(u,v) = \int_B f\left(\sigma\begin{pmatrix} x \\ y \end{pmatrix}\right)\frac{3}{2}\, d(x,y)$$

$$= \frac{3}{2}\int_0^1\int_0^1 2x + \frac{1}{2} + (x+y)^2 \, dx\, dy$$

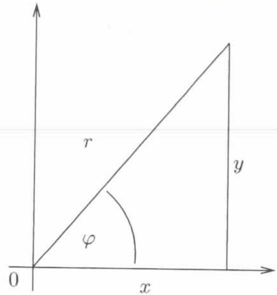
$$= \frac{3}{2}\int_0^1 \frac{3}{2} + \frac{1}{3}(x+y)^3\Big|_{x=0}^{x=1} dy$$

$$= \frac{3}{2}\left(\frac{3}{2} - \frac{1}{12} + \frac{16}{12}\right) = \frac{33}{8}$$

25.5

Polarkoordinaten Der Zusammenhang zwischen den kartesischen Koordinaten (x, y) und den Polarkoordinaten (r, φ) eines Punktes im $\mathbb{R}^2$ ist gegeben durch

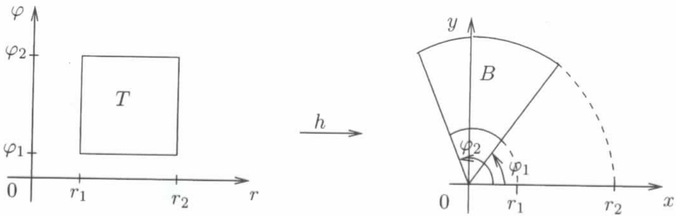
$$x = r\cos\varphi, \quad y = r\sin\varphi.$$



Auf Polarkoordinaten zu transformieren, heißt also, die Substitution

$$(x, y) = h(r, \varphi) = (r\cos\varphi,\, r\sin\varphi) \quad r \geq 0,\ \ \varphi_1 \leq \varphi \leq \varphi_2$$

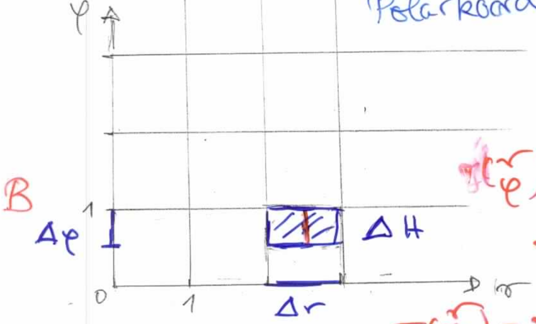
mit $\varphi_2 - \varphi_1 \leq 2\pi$ vorzunehmen. Ist beispielsweise $T = \{(r, \varphi) \in \mathbb{R}^2 : r_1 < r < r_2,\ \varphi_1 < \varphi < \varphi_2\}$ ein Rechteck, so ist $B = h(T)$ ein Teil eines Kreisringes



24.2

(16)

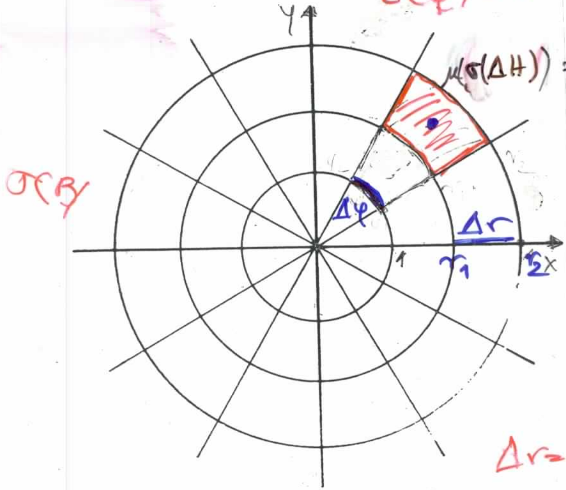
Polarkoordinaten



$$\sigma\begin{pmatrix} r \\ \varphi \end{pmatrix} = r\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \sigma\begin{pmatrix} r \\ \varphi \end{pmatrix}$$

$$\tau\begin{pmatrix} r \\ \varphi \end{pmatrix} = r$$

$$\mu(\sigma(\Delta H)) = r\,\mu(\Delta H)$$



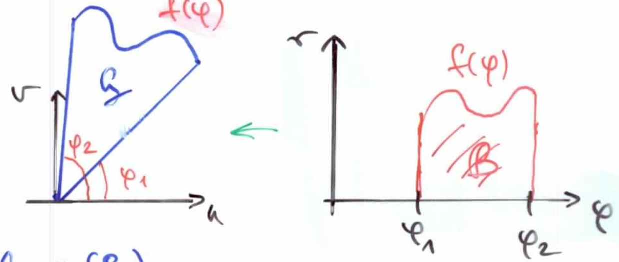
$$\Delta r = r_2 - r_1$$

$$\mu(\sigma(\Delta H)) = \frac{1}{2}\Delta\varphi\, r_2^2 - \frac{1}{2}\Delta\varphi\, r_1^2$$

$$= r\,\Delta\varphi\,\Delta r \qquad r = \frac{1}{2}(r_1 + r_2)$$

3. Binomi

$$G = \left\{ \binom{r}{\varphi} \,\middle|\, \varphi_1 \le \varphi \le \varphi_2 ,\ 0 \le r \le f(\varphi) \right\}$$



$$G = \sigma(B)$$

$$\mu(\sigma(B)) = \int_{\sigma(B)} 1 \, d(u,v)$$

$$= \int_B r \, d(r,\varphi)$$

$$= \int_{\varphi_1}^{\varphi_2} \int_0^{f(\varphi)} r \, dr \, d\varphi$$

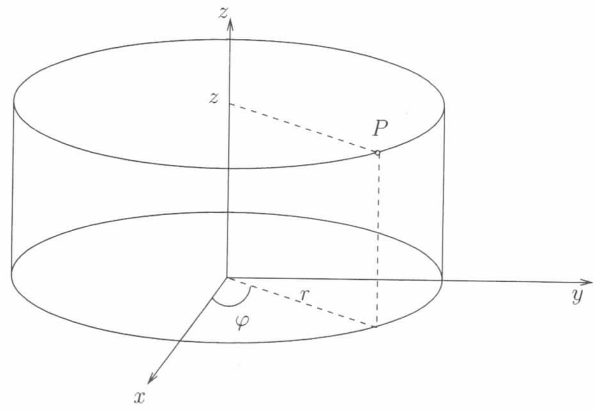
$$= \int_{\varphi_1}^{\varphi_2} \frac{f^2}{2}(\varphi) \, d\varphi \qquad f(\varphi) = \sqrt{\varphi}$$

$$= \int_{\varphi_1}^{\varphi_2} \frac{\varphi}{2} \, d\varphi = \frac{1}{4}(\varphi_2^2 - \varphi_1^2)$$

2.5.6 Zylinderkoordinaten

Die Zylinderkoordinaten (r, φ, z) eines Punktes $P \in \mathbb{R}^3$ mit kartesischen Koordinaten (x, y, z) sind gegeben durch

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z.$$

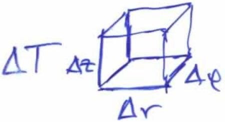


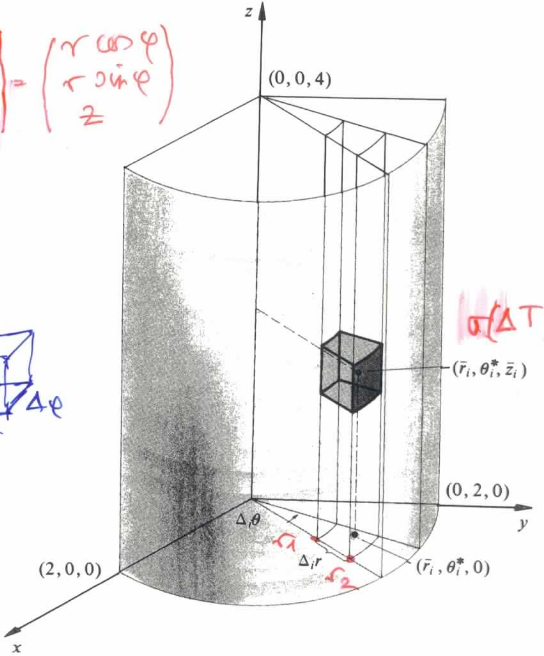
d.h. die Transformationsfunktion ist

$$(x, y, z) = h(r, \varphi, z) = (r\cos\varphi,\ r\sin\varphi,\ z)$$

25.6 Zylinderkoordinaten

$$\sigma\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

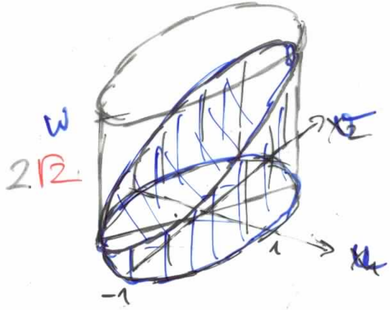




$$\mu(\sigma(\Delta T)) = \bar{r}\,\Delta\varphi\,\Delta r\,\Delta z$$

$$\bar{r} = \frac{1}{2}(r_1 + r_2)$$

$$I\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = r$$



$$G = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \;\middle|\; u^2+v^2 \leq 1,\ 0 \leq w \leq u+v+\sqrt{2} \right\}$$

$$G = \sigma(B) \qquad \sigma\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \tau\begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} = r$$

$$B = \left\{ \begin{pmatrix} r \\ \varphi \\ z \end{pmatrix} \;\middle|\; \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq 2\pi \\ 0 \leq z \leq r\cos\varphi + r\sin\varphi + \sqrt{2} \end{array} \right\}$$

$$\mu(G) = \int_G 1 \, d(u,v,w) = \int_B r \, d(r,\varphi,z)$$

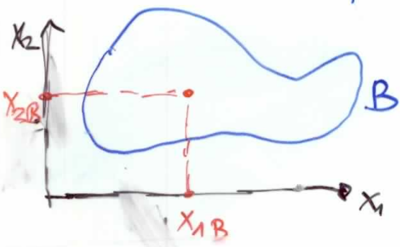
$$= \int_0^1 \int_0^{2\pi} \int_0^{r\cos\varphi + r\sin\varphi + \sqrt{2}} r \, dz\, d\varphi\, dr$$

$$= \int_0^1 \int_0^{2\pi} r^2(\cos\varphi + \sin\varphi) + r\sqrt{2} \, d\varphi\, dr$$

$$= \int_0^1 r^2 \underbrace{\left[\sin\varphi - \cos\varphi\right]_0^{2\pi}}_{0} + 2\pi r\sqrt{2} \, dr$$

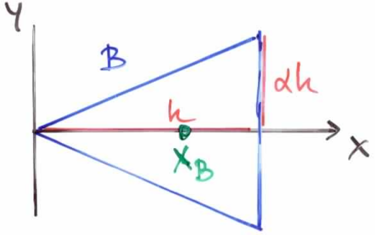
$$= \pi r^2 \sqrt{2}$$

25.6.3 Schwerpunkt



$$x_{iB} = \frac{1}{\mu(B)} \int_B x_i \, dx$$

Beispiel

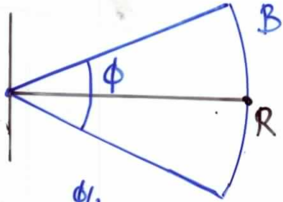


$$X_B = \frac{1}{\alpha h^2} \int_B x \, dx$$

$$= \frac{1}{\alpha h^2} \int_0^h \int_{-\alpha x}^{\alpha x} x \, dy \, dx$$

$$= \frac{1}{\alpha h^2} \int_0^h 2\alpha x^2 \, dx = \frac{1}{\alpha h^2} \frac{2}{3} \alpha h^3$$

$$= \frac{2}{3} h$$



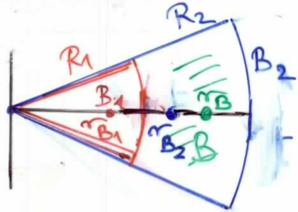
$$\int_B x \, d(x,y) \qquad (21)$$

$$= \int_{-\phi/2}^{\phi/2} \int_0^R r \cos\varphi \; r \, dr \, d\varphi$$

$$= \int_{-\phi/2}^{\phi/2} \frac{1}{3} R^3 \cos\varphi \, d\varphi = \frac{1}{3} R^3 \sin\varphi \Big|_{-\phi/2}^{\phi/2} = \frac{2}{3} R^3 \sin \phi/2$$

$$\mu(B) = \frac{1}{2} R^2 \phi$$

$$x_B = r_B = \frac{4}{3} R \underbrace{\frac{1}{\phi} \sin \phi/2}_{\to \frac{2}{3} R}$$



Hebel

$$(r_{B_2} - r_{B_1}) \mu(B_1)$$
$$= (r_B - r_{B_2}) \mu(B)$$

$$r_B = \frac{\mu(B_1)}{\mu(B)} (r_{B_2} - r_{B_1}) + r_{B_2}$$

$$= \frac{R_1^2}{R_2^2 - R_1^2} \frac{4}{3} \frac{1}{\phi} \sin \frac{\phi}{2} (R_2 - R_1) + \frac{4}{3} R_2 \frac{1}{\phi} \sin \frac{\phi}{2}$$

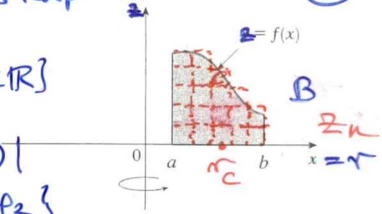
$$= \frac{4}{3} \frac{1}{\phi} \sin \frac{\phi}{2} \left(\frac{R_1^2}{R_2 + R_1} + R_2 \right) \geq R_1$$

für $\phi \to 0$

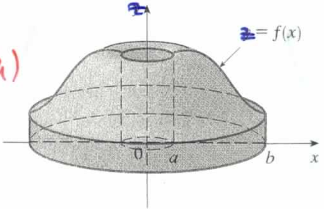
15.6.4 Rotationskörper

$B \subseteq \{(r,z) \mid r \geq 0,\ z \in \mathbb{R}\}$

$D = \{(r\cos\varphi,\ r\sin\varphi,\ z) \mid$
$(r,z) \in B,\ \varphi_1 \leq \varphi \leq \varphi_2\}$



$\mu(D) = r_B \cdot \mu(B)(\varphi_2 - \varphi_1)$



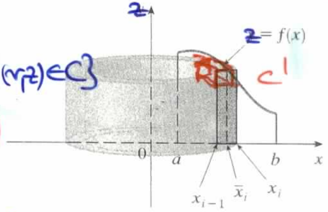
Bew

$$r_B = \frac{1}{\mu(B)} \int_B r \, d(r,z)$$

$$= \frac{1}{\mu(B)} \lim_{n\to\infty} r_C \, \mu(C)$$

FIGURE 3

$\varphi_1 = 0,\ \varphi_2 = \frac{\pi}{4}$

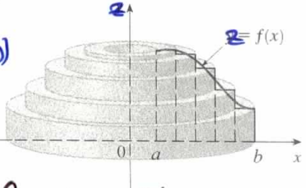


$C' = \{(r\cos\varphi,\ r\sin\varphi,\ z) \mid (r,z) \in C\}$

$\mu(C') = r_C \, \mu(C)(\varphi_2 - \varphi_1)$

$\rightsquigarrow$ Zerlegung von D

$O(Z_n', D) - U(Z_n', D) \to 0$



$(\varphi_2 - \varphi_1) \sum r_C \, \mu(C) \to \mu(D)$

$\to (\varphi_2 - \varphi_1) \, r_B \, \mu(B)$

Z_n' nicht geeignet für $\int$ da Weite $\not\to 0$

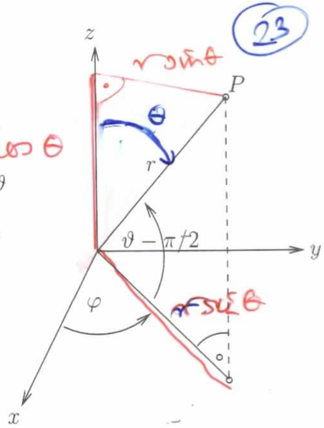
25.7

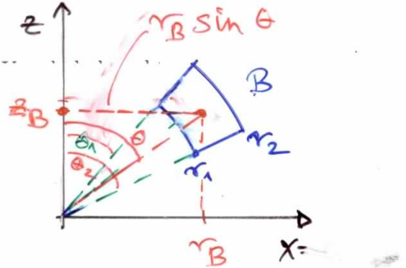
Kugelkoordinaten Die Kugelkoordinaten (r, ϑ, φ) sind mit den kartesischen Koordinaten (x, y, z) verknüpft durch

$$x = r\cos\varphi\,\sin\vartheta,\ \ y = r\sin\varphi\,\sin\vartheta,\ \ z = r\cos\vartheta$$

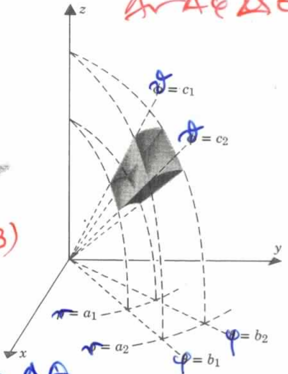
wobei $r \geq 0$, $0 \leq \vartheta \leq \pi$ und $0 \leq \varphi \leq 2\pi$. Die Substitutionsfunktion ist

$$(x, y, z) = \sigma(r, \varphi, \vartheta)$$
$$(r\cos\varphi\,\sin\vartheta,\ r\sin\varphi\,\sin\vartheta,\ r\cos\vartheta)$$





$\Delta r\,\Delta\varphi\,\Delta\theta$



a spherical wedge

$$\mu(D) = (r_B \sin\theta_B\ \Delta\varphi)\,\mu(B)$$

$$\mu(B) = \frac{r_1+r_2}{2}\,\Delta r\,\Delta\theta$$

$$\mu(D) = \bar{r}^2 \sin\theta_B\,\Delta r\,\Delta\varphi\,\Delta\theta$$

$$\bar{r} = \sqrt{r_B \cdot \frac{r_1+r_2}{2}}$$

$$\boxed{\tau(r, \varphi, \theta) = r^2 \sin\theta}$$

$G =$ Kugel um 0

Radius R

σ Kugelkoordinaten

$G = \sigma(B) \qquad B = [0,R] \times [0,2\pi] \times [0,\pi]$

$\tau = r^2 \sin\theta$

$$\mu(G) = \int_G 1$$

$$= \int_B r^2 \sin\theta \; d(r,\varphi,\theta)$$

$$= \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, d\theta \, d\varphi \, dr$$

$$= \int_0^R \int_0^{2\pi} \left[-r^2 \cos\theta\right]_0^{\pi} d\varphi \, dr$$

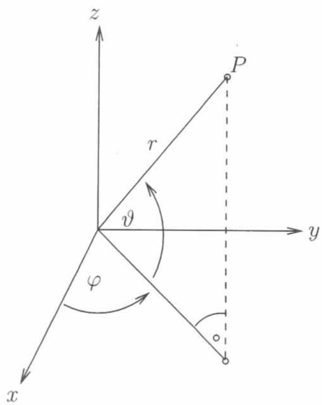
$$= \int_0^R \int_0^{2\pi} 2r^2 \, d\varphi \, dr$$

$$= \int_0^R 4r^2\pi \, dr$$

$$= \frac{4}{3} R^3 \pi$$

25.7 Kugelkoordinaten

Kugelkoordinaten: nicht für Ingenieure Die Kugelkoordinaten (r, ϑ, φ) sind mit den kartesischen Koordinaten (x, y, z) verknüpft durch



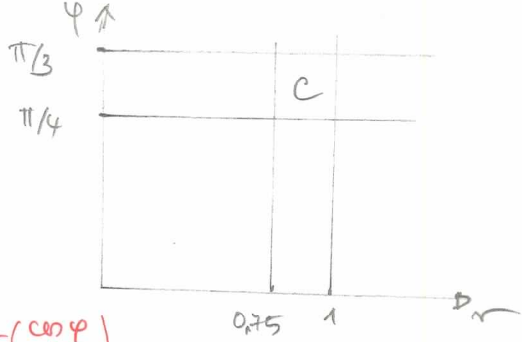
$$x = r\cos\vartheta\cos\varphi,\ y = r\cos\vartheta\sin\varphi,\ z = r\sin\vartheta,$$

wobei $r \geq 0$, $-\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}$ und $0 \leq \varphi < 2\pi$. Die Transformationsfunktion ist

$$\begin{aligned}(x, y, z) &= h(r, \vartheta, \varphi)\\ &= (r\cos\vartheta\cos\varphi,\ r\cos\vartheta\sin\varphi,\ r\sin\vartheta)\end{aligned}$$

Das

Kugelvolumen mit Radius R ist $\frac{2}{3}\, R^3 \int_0^{2\pi} d\varphi = \frac{4}{3}\,\pi R^3$.



$$\sigma\begin{pmatrix} r \\ \varphi \end{pmatrix} = r\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$p = \begin{pmatrix} 3/4 \\ \pi/4 \end{pmatrix} \qquad \left\{ J\sigma(p)\begin{pmatrix} \Delta r \\ \Delta\varphi \end{pmatrix} + p \;\middle|\; \begin{matrix} 0 \le \Delta r \le 1/ \\ 0 \le \Delta\varphi \le \pi/12 \end{matrix} \right.$$