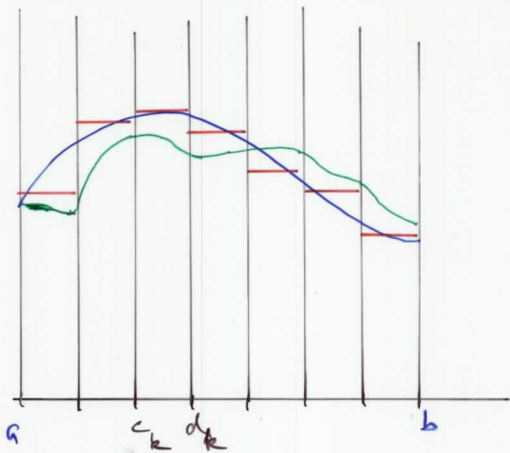


# Bes Thm 23.9 (Summation)



$$| \text{blue polygon} - \text{red rectangle} | \leq \varepsilon \cdot \mu[c, d]$$

$$| \text{green polygon} - \text{red rectangle} | \leq \varepsilon \cdot \mu[c, d]$$

$$| W([a, b]) - \int_{[a, b]} f |$$

$$\leq \sum | W([c_k, d_k]) - \int_{[c_k, d_k]} f | \leq 2\varepsilon \mu([a, b])$$

# Fubini

and examples

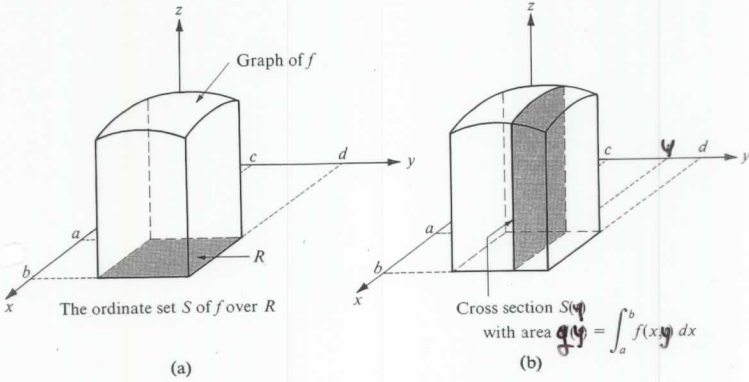
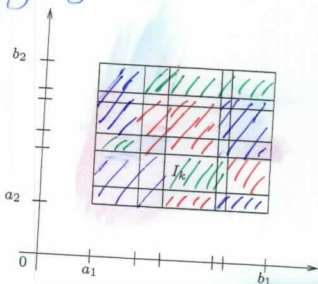


FIGURE 2.4 The volume of  $S$  is the integral of the cross-sectional area:  $V(S) = \int_c^d g(y) dy$

# 23 1.2. Zerlegungen

(5)

Weite  $|z|$   
 $= \max |f|$   
 $I$  Teilintervall



Gitter-  
Zerlegung

$$f_n \approx f$$

$$g(y) = \int_I f(x,y) dx$$

$$\int f_n = \sum_c \left( \sum_k f_n(I_k \times I_c) \mu(I_k) \right) \mu(I_c)$$

$$\approx \int g(y) dy$$

$$\approx \int g(y) dy$$

Fubini

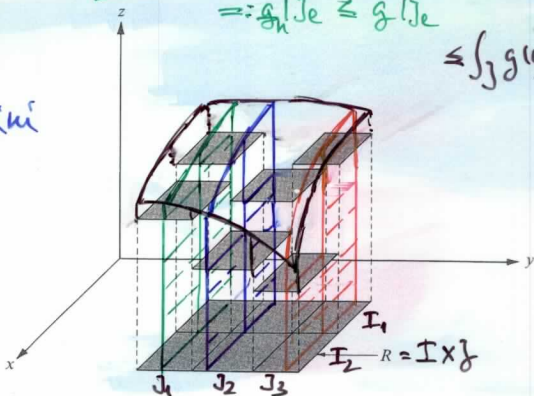
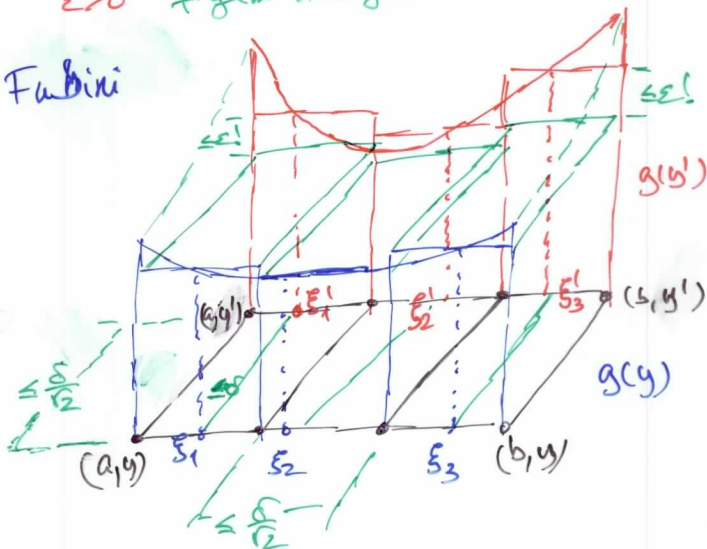


FIGURE 2.8 The graph of a step function defined over a rectangle  $R$ .

$\varepsilon > 0$   $f$  glatt stetig  $\leadsto \delta > 0$  (23)

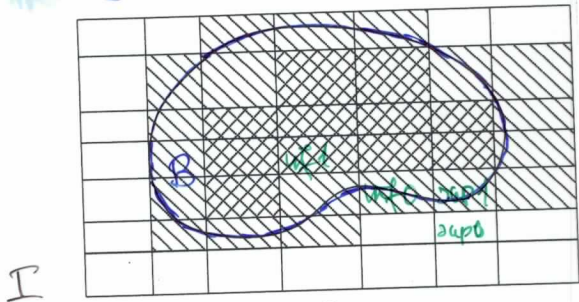
Fubini



$\Rightarrow |g(y') - g(y)| \leq \varepsilon(b-a)$   
Mittelwert

$\Rightarrow g(y)$  stetig

## 2.4.1 Jordan messbare Mengen ④



$B \subseteq \mathbb{R}^n$  beschränkt

innerer Inhalt 
$$U(I, Z, B) = \sum_{J \in Z, J \subseteq B} \mu(J)$$

äußerer Inhalt 
$$\leq O(I, Z, B) = \sum_{J \in Z, J \cap B \neq \emptyset} \mu(J)$$

$B$  (Jordan) messbar

$\Leftrightarrow \exists \frac{I}{\mu(B)}, Z_n \lim_{n \rightarrow \infty} O(I, Z_n, B) - U(I, Z_n, B) = 0$

Maß 
$$\mu(B) = \lim_{n \rightarrow \infty} U(I, Z_n, B)$$



↔ charakteristische Funktion

$$\chi_B: I \rightarrow \mathbb{R} \quad \chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

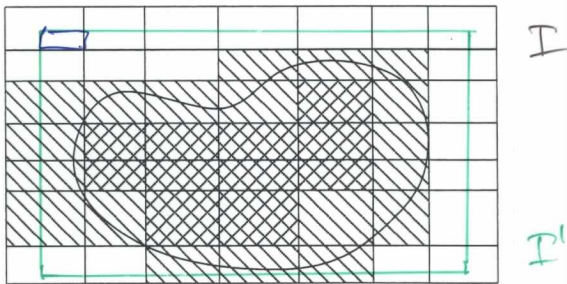
integrierbar,  $\mu(B) = \int_I \chi_B(x) dx$

Folgerung: unabh. von  $Z_n$

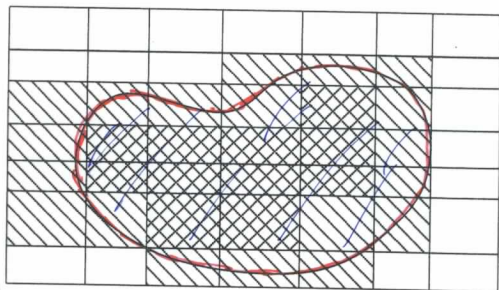
jede Folge  $Z_n$  mit Weite  $Z_n \rightarrow 0$

Behauptung: unabh. von  $I$

Beweis  $\forall B \subset I' \subseteq I$



(-2)



Rand  
 $\partial B$

$B$

Kor. 24.2  $B \subseteq \mathbb{R}^n$  beschränkt

$B$  messbar  $\Leftrightarrow \partial B$  messbar

$$\mu(\partial B) = 0$$

Def  $B$  Nullmenge

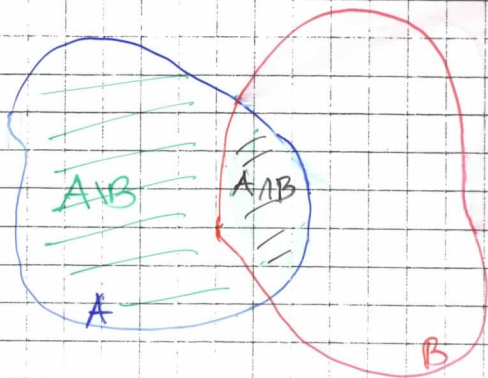
$$\Leftrightarrow \mu(B) = 0$$

Bew  $\forall C \in \mathcal{Z}$ ,  $\mathbb{Z}$ -Zahl  $m \in \mathbb{I}$  entweder  
 $C \subseteq B$  od  $C \cap B = \emptyset$  od  $C \cap \partial B \neq \emptyset$

falls  $B$  kompakt

24.12

(-1)



Satz:  $A, B$  messbar

$\rightarrow A \cup B, A \cap B, A \setminus B$   
messbar

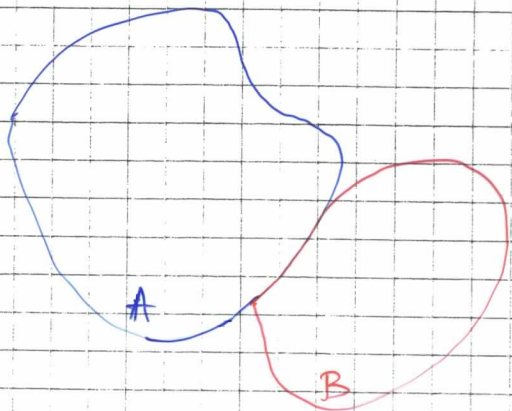
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

$$\mu(A) = \mu(A \setminus B) + \mu(A \cap B)$$

Beweis. Warte  $Z_n \rightarrow 0$

für  $A \cap B$ : doppelt gezählt





$A, B$  nicht überlappend

$$\Leftrightarrow A \cap B = \emptyset \Rightarrow \mu(A \cap B) = 0$$

$$\rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$$

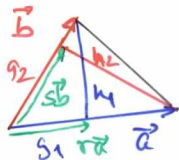
Bew.  $\mu(A \cap B) = 0$

$n=2$

①  $\vec{a}_1 \perp \vec{a}_2 \Rightarrow |\det(\vec{a}_1, \vec{a}_2)| = \|\vec{a}_1\| \cdot \|\vec{a}_2\|$

Beweis  $\vec{a}_c = \|\vec{a}_c\| \vec{b}_c$ ,  $\det A = \|\vec{a}_1\| \cdot \|\vec{a}_2\| \det B$   
 $\vec{b}_1, \vec{b}_2$  ON-Basis  $\Rightarrow |\det B| = 1$

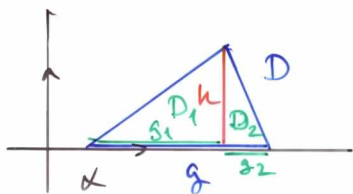
② Im Dreieck mit Grundseite  $g_i$   
und Höhe  $h_i$  gilt  
 $g_1 h_1 = g_2 h_2$



Bew.  $g_1 h_1 =$

$$|\det(\vec{a}, \vec{b} - r\vec{a})| = |\det(\vec{a}, \vec{b})| \\ = |\det(\vec{a} - s\vec{b}, \vec{b})| = g_2 h_2$$

③

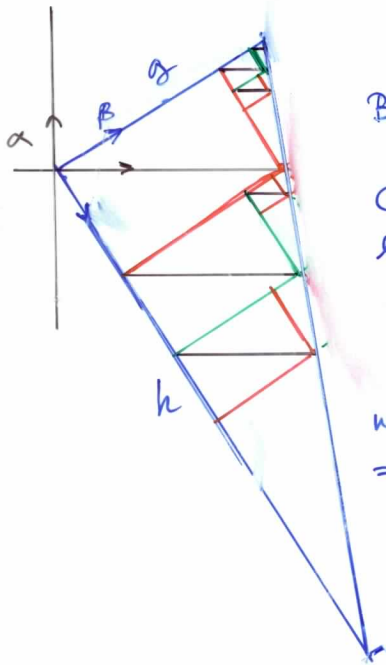


$$\mu_x(D) = \frac{1}{2} g h$$

Bew:  $\mu_x(D_i) = g_i h$  (Übung)

$$\mu_x(D) = \mu_x(D_1) + \mu_x(D_2) \\ = \frac{1}{2} g_1 h_1 + \frac{1}{2} g_2 h_2 = \frac{1}{2} g h$$

④ D rechtw. Dreieck  $\Rightarrow \mu_x(D) = \frac{1}{2} gh$



Bew



C zelle für  $x$   
 $\mu_x(C) = \mu_p(C)$

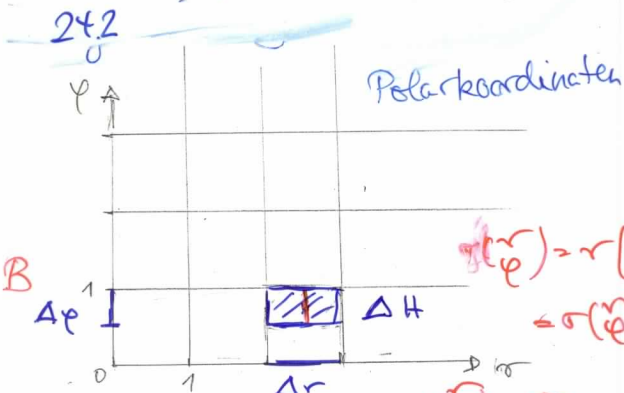
$$\mu_x(D) = \lim_{n \rightarrow \infty} \sum_{C \in Z_n} \mu_x(C)$$

$$= \lim_{n \rightarrow \infty} \sum_{C \in Z_n} \mu_p(C)$$

$$= \mu_p(D) \\ = \frac{1}{2} gh$$

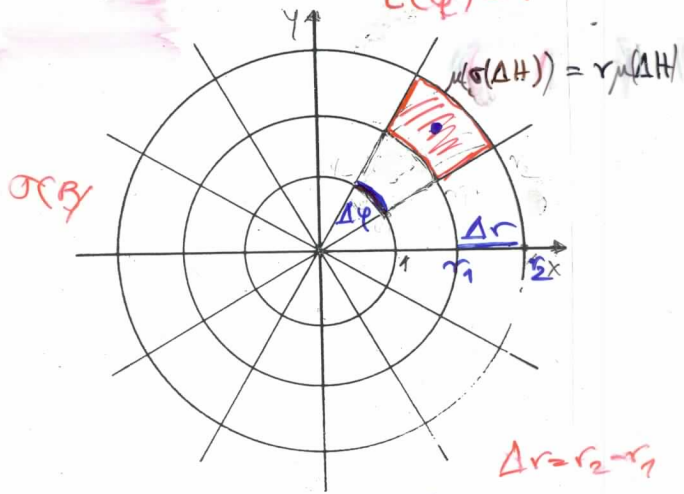
⑤ R Rechteck  
 $\Rightarrow \mu_x(R) = gh$

⑥  $\mu_x = \mu_p$



$\sigma(r, \varphi) = r \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$   
 $= \sigma(r, \varphi)$

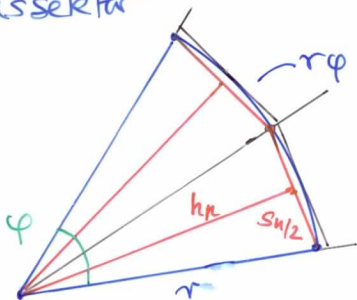
$\sigma(\varphi) = r$



$\mu(\sigma(\Delta H)) = \frac{1}{2} \Delta \varphi r_2^2 - \frac{1}{2} \Delta \varphi r_1^2$   
 $= r \Delta \varphi \Delta r \quad r = \frac{1}{2} (r_1 + r_2)$

3. Bikoni

Kreissektor



Zerlegung

$z_n$

$n$  Dreiecke

$k_n \cdot s_n / 2$

$n$  Trapeze

$$s_n(r - k_n) + \frac{s_n}{2k_n}(r - k_n)^2$$

$$O(z_n, B) - U(z_n, B) =$$

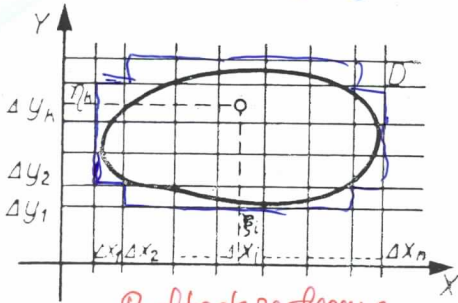
$$\underbrace{n \cdot s_n}_{r\varphi} (r - k_n) + \frac{n \cdot s_n}{2k_n} (r - k_n)^2 \rightarrow 0$$

$\downarrow 0$        $\downarrow 0$

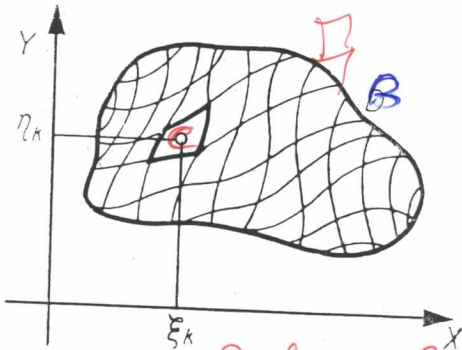
$$U(z_n, B) = n k_n \frac{s_n}{2}$$

$$= \frac{1}{2} \underbrace{k_n}_{dr} \underbrace{n s_n}_{r\varphi} \rightarrow \frac{1}{2} r^2 \varphi$$

# Zerlegungen



Rechteckzerlegung  
sogar Gitterzerlegung



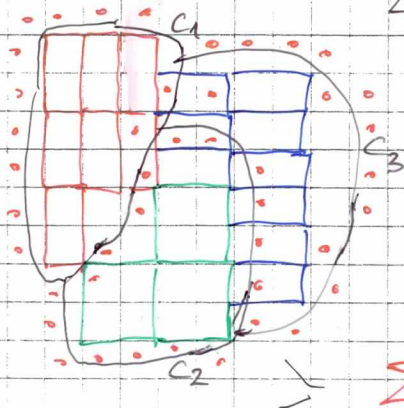
Zerlegung Z

$C \in Z$

$B \subseteq \cup C$  bedeckt  
 ~~$C \in Z$~~

(7)

$$Z = C_1 \cup C_2 \cup C_3$$



$$\sum \square \cdot < \epsilon$$

Lemma Zu jeder Zerlegung  $Z$  von  $B$   
 und  $\forall \epsilon > 0 \exists$  Gitterzerlegung  $Z^\epsilon$

$$\sum_{\substack{J \in Z^\epsilon - Z, \\ J \cap B \neq \emptyset}} \mu(J) < \epsilon$$

$\exists \neq C$  alle  $C \in Z$

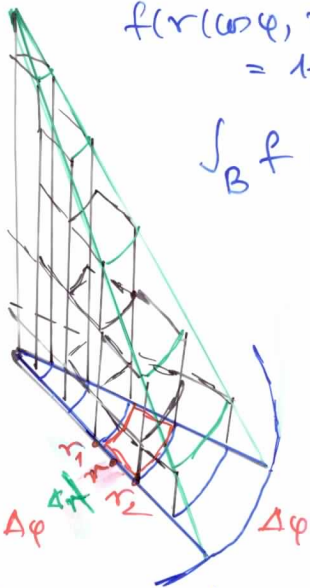
Bew  $Z_C$  Gitterzerlegung von  $C \in Z$   
 $O(Z_C, C) - U(Z_C, C) < \epsilon / |Z|$

B = Einheitskreis

(12)

$$f(r(\cos \varphi, \sin \varphi)) = 1-r$$

$$\int_B f \, d(x,y) = ?$$



$$\mu(F)$$

$$= \frac{1}{2} (r_1 + r_2) \Delta r \Delta \varphi$$

$$z_n \quad \Delta r = \frac{1}{n}, \quad \Delta \varphi = \frac{1}{n} 2\pi$$

$$r_1 = \frac{k}{n}$$
$$r_2 = \frac{k+1}{n}$$

$$U(z_n, f) = \sum_{C \in Z_n} \min f(C) \mu(C)$$

$$= n \sum_{k=0}^{n-1} \left(1 - \frac{k+1}{n}\right) \frac{1}{2} \frac{2k+1}{n} \cdot \frac{1}{n} \cdot \frac{2\pi}{n}$$

r



(13)

$$\frac{\pi}{n^2} \left( \sum_{k=0}^{n-1} 2k+1 - \frac{1}{n} \sum_{k=0}^{n-1} (k+1)(2k+1) \right)$$

$$= \frac{\pi}{n^2} \left( 2 \sum_{k=0}^{n-1} k + n - \frac{1}{n} \left( 2 \sum_{k=0}^{n-1} k^2 + 3 \sum_{k=0}^{n-1} k + n \right) \right)$$

$$= \pi \left( \frac{n(n-1)}{n^2} + \frac{1}{n} - \frac{2(n-1)n(2n-1)}{6n^3} - \frac{3n(n-1)}{2n^3} + \frac{1}{n^2} \right)$$

$$\rightarrow \pi \left( 1 - \frac{4}{6} \right) = \frac{1}{3} \pi$$

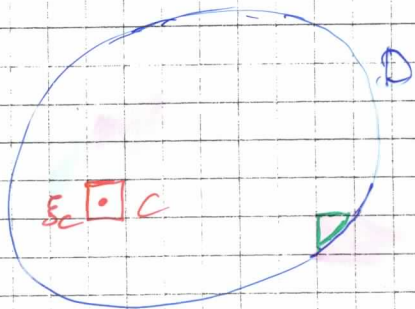
$$\eta > 0$$

$$M = \max |f(x)|$$

$$\varepsilon < \eta/27$$

$$\varepsilon M < \eta/27 \mu(D)$$

(18)



$\sum$  Randkästchen  
 $< \varepsilon$

$$f(\xi_C) \mu(C) \pm \varepsilon \mu(C)$$

$$M \cdot \mu(C)$$

$$\int_C f(x) dx \pm 2\varepsilon \mu(C)$$

$\sum$

$$\int_D f(x) dx \pm \varepsilon M \pm 2\varepsilon \mu(C) \pm \varepsilon M$$

$\pm \eta \mu(D)$