Satz 16.3

Hieber VIII 1.3

AnaIIss08.pdf

$F: U \to \mathbb{R}^n$, $\vec{p} \in U \subseteq \mathbb{R}^n$

F stetig diffbar

$J_F(\vec{p})$ invertierbar

$\Leftrightarrow \exists\ \vec{p} \in U' \subseteq U$, $V' \subseteq \mathbb{R}^n$

$g: V' \to U'$ stet diffbar

$F(\vec{x}) = \vec{y} \Leftrightarrow g(\vec{y}) = \vec{x} \quad \forall \vec{x} \in U'$
$\forall \vec{y} \in V'$

Zusatz $J_F(\vec{x})$ invertierbar für $\vec{x} \in U'$

$$J_g(\vec{y}) = J_F(\vec{x})^{-1} \quad \vec{y} = F(\vec{x})$$

Beisp $\begin{pmatrix} r \\ \varphi \end{pmatrix} \overset{F}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$

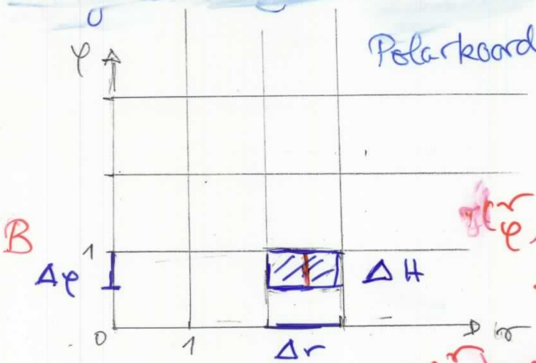
$\begin{pmatrix} \sqrt{x^2+y^2} \\ \arcsin \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix} \overset{g}{\longleftarrow}$

$$J_F = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$U' = \left\{ \begin{pmatrix} r \\ \varphi \end{pmatrix} \middle| \begin{array}{c} 0 < r \\ -\pi < \varphi < \pi \end{array} \right\} \quad V' = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x > 0 \right\}$

Bew $\Phi_y(x) = y + x - F(x) \quad g(y) = x \Leftrightarrow \Phi_y(x) = x$

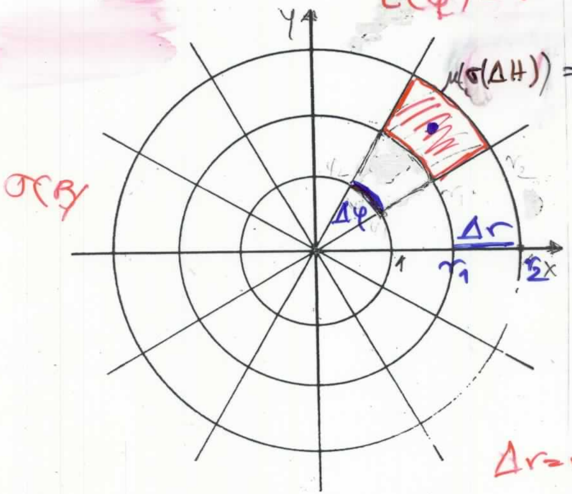
Polarkoordinaten



$$\sigma\begin{pmatrix} r \\ \varphi \end{pmatrix} = r\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$
$$= \sigma\begin{pmatrix} r \\ \varphi \end{pmatrix}$$

$$\tau\begin{pmatrix} r \\ \varphi \end{pmatrix} = r$$

$$\mu(\sigma(\Delta H)) = r\,\mu(\Delta H)$$



$$\Delta r = r_2 - r_1$$

$$\mu(\sigma(\Delta H)) = \frac{1}{2}\Delta\varphi\, r_2^2 - \frac{1}{2}\Delta\varphi\, r_1^2$$
$$= r\,\Delta\varphi\,\Delta r \qquad r = \frac{1}{2}(r_1 + r_2)$$

3. Binomi

18.4

$\vec{x}(t)$ $t \in [a,b]$ stetig differenzierbar

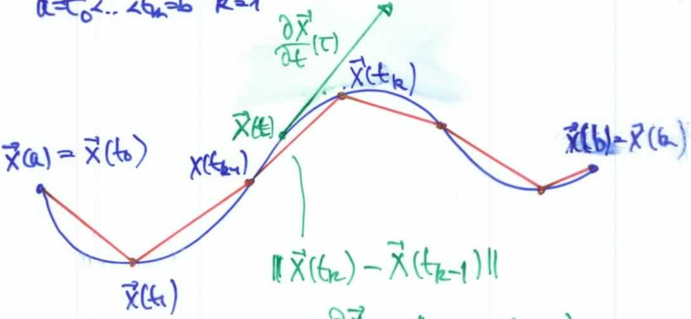
$$\frac{\partial \vec{x}}{\partial t}(t) = \begin{pmatrix} \partial x_1 / \partial t \\ \vdots \\ \partial x_n / \partial t \end{pmatrix}(t) \quad \text{stetig}$$

$$\left\|\frac{\partial \vec{x}}{\partial t}\right\| > 0$$

Weglänge $L(\vec{x}_{[a,b]}) = \int_a^b \left\|\frac{\partial \vec{x}}{\partial t}\right\| dt$

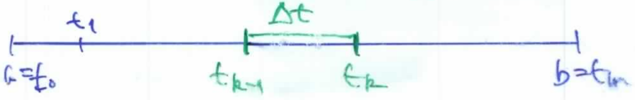
$$= \int_a^b \sqrt{\left(\frac{\partial x_1}{\partial t}\right)^2 + \dots + \left(\frac{\partial x_n}{\partial t}\right)^2}\, dt$$

$$= \sup_{a=t_0 < \dots < t_m = b} \sum_{k=1}^{m} \|\vec{x}(t_k) - \vec{x}(t_{k-1})\|$$

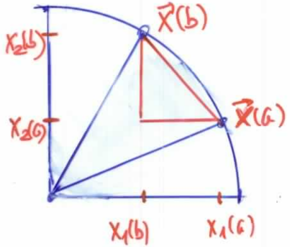


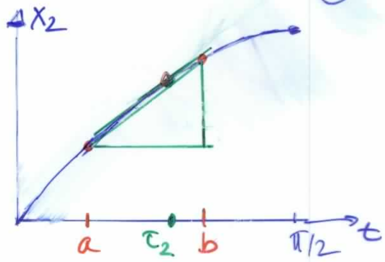
$$\|\vec{x}(t_k) - \vec{x}(t_{k-1})\|$$

$$\approx \left\|\frac{\partial \vec{x}}{\partial t}(\tau)\right\| \underbrace{(t_k - t_{k-1})}_{\Delta t}$$

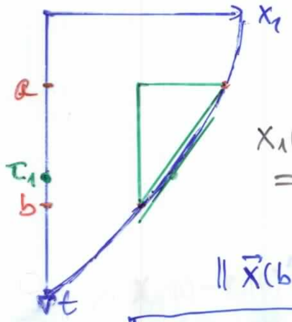


Mittelwertsatz





$$x_2(a) - x_2(b) = \frac{\partial x_2}{\partial t}(\tau_2)(b-a)$$



$$x_1(a) - x_1(b) = \frac{\partial x_1}{\partial t}(\tau_1)(b-a)$$

$$\begin{aligned}
&\|\vec{x}(b) - \vec{x}(a)\| \\
&= \sqrt{[x_1(b) - x_1(a)]^2 + [x_2(b) - x_2(a)]^2} \\
&= \sqrt{\left[\frac{\partial x_1}{\partial t}(\tau_1)(b-a)\right]^2 + \left[\frac{\partial x_2}{\partial t}(\tau_2)(b-a)\right]^2} \\
&= \sqrt{\frac{\partial x_1}{\partial t}(\tau_1)^2 + \frac{\partial x_2}{\partial t}(\tau_2)^2}\,(b-a)
\end{aligned}$$

$$\left\|\frac{\partial \vec{x}}{\partial t}\right\| = 1 \implies \searrow 1 \quad \text{für } b-a \to 0$$

Lemma 18.5 $\vec{x}:[a,b]\to\mathbb{R}^n$ stetig diffbar

$\left\|\frac{\partial\vec{x}}{\partial t}\right\| = 1 \implies$

$$b-a = \sup \underbrace{\sum_{k=1}^{m} \|\vec{x}(t_k)-\vec{x}(t_{k-1})\|}_{=:L}$$

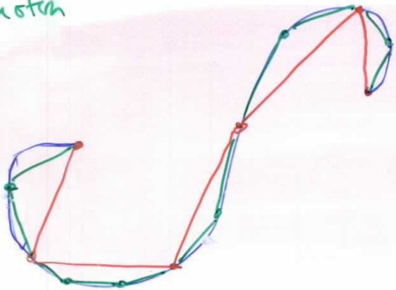
$a=t_0<t_1<\dots<t_m=b$

Bew.

$$L = \sum_{k=1}^{m} \underbrace{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial x_i}{\partial t}(\tau_{ik})\right)^2}}_{\to 1} (t_k-t_{k-1})$$

$$L \to \sum_{k=1}^{m}(t_k-t_{k-1}) = b-a \quad \square$$

monoton



Korollar 18.7 $\vec{x}(t)$ $t \in [a,b]$ stetig diffbar

$$\left\|\frac{\partial \vec{x}}{\partial t}\right\| > 0 \rightarrow L(\vec{x}_{[a,b]}) = \int_a^b \left\|\frac{\partial \vec{x}}{\partial t}\right\| dt$$

Beweis $s = s(t) = \int_a^t \left\|\frac{\partial \vec{x}}{\partial t}(\tau)\right\| d\tau$

$$\frac{\partial s}{\partial t} = \left\|\frac{\partial \vec{x}}{\partial t}(t)\right\| > 0$$

$t = t(s)$ Umkehrfunktion

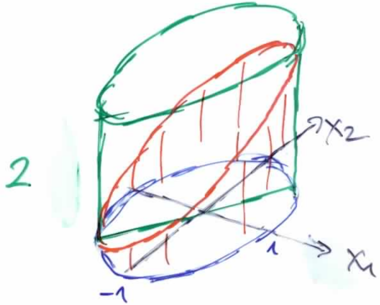
$$\frac{\partial \vec{x}}{\partial s} = \frac{\partial \vec{x}}{\partial t}\frac{\partial t}{\partial s} = \frac{\partial \vec{x}}{\partial t}\frac{1}{\frac{\partial s}{\partial t}} = \frac{\partial x}{\partial t}\frac{1}{\left\|\frac{\partial \vec{x}}{\partial t}\right\|}$$

$$\Rightarrow \left\|\frac{\partial \vec{x}}{\partial s}\right\| = 1$$

$$L(\vec{x}_{[a,b]}) = L(\vec{x}(t(s))_{[s(a),s(b)]}$$

$$= s(b) - \underbrace{s(a)}_{=0} = \int_a^b \left\|\frac{\partial \vec{x}}{\partial t}\right\| dt$$

19.1 Wegintegral im Skalarenfeld



$$\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$$

$$f(\vec{x}) = x_1 + x_2 + \sqrt{2} \qquad \left\| \frac{\partial \vec{x}}{\partial t} \right\| = 1$$

$$\int_0^{2\pi} f(\vec{x}(t))\,dt \quad = \text{IIII}$$

$$= \int_0^{2\pi} \cos t + \sin t + \sqrt{2}\,dt$$

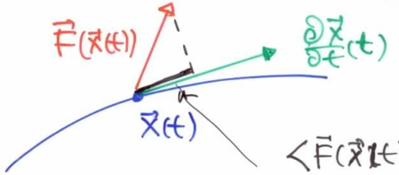
$$= 2\sqrt{2}\,\pi$$

Vektorfeld $\vec{F}(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{pmatrix}$

(Kurven)Integral entlang $\Gamma: \vec{x}(t) \quad t \in [a,b]$

$$\int_\Gamma \vec{F} \cdot d\vec{x} = \int_a^b \left\langle \vec{F}(\vec{x}(t)) \,\middle|\, \frac{\partial \vec{x}}{\partial t}(t) \right\rangle dt$$

$$= \int_\Gamma \sum_i F_i \, dx_i$$



$\left\langle \vec{F}(\vec{x}(t)) \,\middle|\, \frac{\partial \vec{x}}{\partial t}(t) \right\rangle$

falls $\left\| \frac{\partial \vec{x}}{\partial t}(t) \right\| = 1$

$$= \sum_{i=1}^n \int F_i(\vec{x}(t)) \cdot \frac{\partial x_i}{\partial t}(t) \, dt$$

$\int_\Gamma \vec{F} \cdot d\vec{x}$ unabhängig von Parametrisierung

$s = s(t)$, $t = t(s)$,

$$\int_{s(a)}^{s(b)} \left\langle \vec{F}(\vec{x}(t(s))) \,\middle|\, \frac{\partial \vec{x}}{\partial s} \right\rangle ds$$

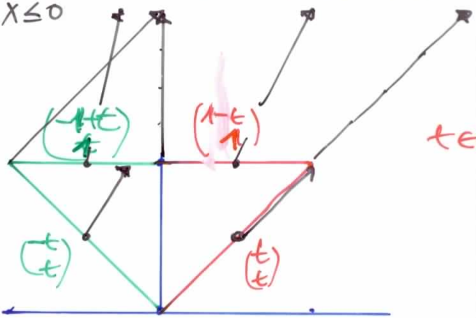
$$= \int_a^b \left\langle \vec{F}(\vec{x}(t)) \,\middle|\, \frac{\partial \vec{x}}{\partial t} \right\rangle \frac{\partial t}{\partial s} \, ds$$

$$= \int_a^b \left\langle \vec{F}(\vec{x}(t)) \,\middle|\, \frac{\partial \vec{x}}{\partial t} \right\rangle dt$$

$$F\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases}\begin{pmatrix}x\\y\end{pmatrix} & x \geq 0\\ \begin{pmatrix}x^2\\y\end{pmatrix} & x \leq 0\end{cases}$$

19.3 Wegintegral im Vektorfeld

8



$t \in [0,1]$

$t \in [0,1]$

$$\int_0^1 \left\langle \begin{pmatrix}t\\t\end{pmatrix} \middle| \begin{pmatrix}1\\1\end{pmatrix} \right\rangle dt + \int_0^1 \left\langle \begin{pmatrix}1-t\\1\end{pmatrix} \middle| \begin{pmatrix}-1\\0\end{pmatrix} \right\rangle dt$$

$2t \qquad t-1$

$$= \int_0^1 3t-1 \, dt = \frac{1}{2}$$

$$\int_0^1 \left\langle \begin{pmatrix}(-t)^2\\t\end{pmatrix} \middle| \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle dt + \int_0^1 \left\langle \begin{pmatrix}(-1+t)^2\\1\end{pmatrix} \middle| \begin{pmatrix}1\\0\end{pmatrix} \right\rangle dt$$

$$= \int_0^1 -t^2 + t + (t-1)^2 \, dt$$

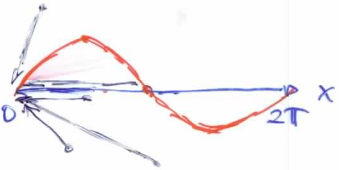
$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{3} = -\frac{1}{6}$$

19.3 Wegintegral im Vektorfeld

$F(\vec{x}) \in \mathbb{R}^n$, $\quad \vec{x}(t) \in \mathbb{R}^n \quad t \in [a,b]$

$$\int_a^b \left\langle F(\vec{x}(t)) \,\middle|\, \frac{\partial \vec{x}}{\partial t}(t) \right\rangle dt$$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$



$\vec{x}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad t \in [0, 2\pi]$

$\vec{x}(t) = \begin{pmatrix} t \\ \sin t \end{pmatrix} \quad t \in [0, 2\pi]$

$$\frac{\partial \vec{x}}{\partial t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \int_0^{2\pi} \left\langle -\begin{pmatrix} t \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle dt = \int_0^{2\pi} -t \, dt = -2\pi^2$$

$$\frac{\partial \vec{x}}{\partial t} = \begin{pmatrix} 1 \\ \cos t \end{pmatrix} \quad \int_0^{2\pi} \left\langle -\begin{pmatrix} t \\ \sin t \end{pmatrix} \middle| \begin{pmatrix} 1 \\ \cos t \end{pmatrix} \right\rangle dt$$

$$= \int_0^{2\pi} -t - \sin t \cos t \, dt = \left[-\frac{1}{2} t^2 - \frac{1}{2} \sin^2 t \right]_0^{2\pi} = -2\pi^2$$

$$\vec{x}(t) = \begin{pmatrix} 2t \\ 0 \end{pmatrix} \quad \frac{\partial \vec{x}}{\partial t} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\int_0^{\pi} \left\langle -\begin{pmatrix} t \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\rangle dt = \int_0^{\pi} -2t \, dt = -2\pi^2$$

20.1 $M \subseteq \mathbb{R}^n$ offen, $F: M \to \mathbb{R}^n$

$\varphi: M \to \mathbb{R}$ stetig part. diffbar

$F(p) = \operatorname{grad}\varphi(p) \quad \forall p \in M$

φ Potential / Stammfunktion für F

Γ: $\vec{x}(t) \in M \quad t \in [a,b]$ stetig diffbar

$$\Rightarrow \int_\Gamma F \cdot d\vec{x} = \int_a^b \langle F(\vec{x}(t)) \mid \frac{\partial \vec{x}}{\partial t} \rangle \, dt$$

$$= \varphi(\vec{x}(b)) - \varphi(\vec{x}(a))$$

Wegunabhängigkeit

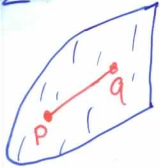
Folg. $\varphi + C$ eindeutig bis auf C

Folg Sind die F_i stetig diffbar

so $$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

$$\text{Schwarz} \Rightarrow \frac{\partial^2 \varphi}{\partial x_j \partial x_i} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

20.2



konvex $\forall p, q$

$\Downarrow$

sternförmig
$\exists p \, \forall q$

$\Downarrow$

Wegzusammenhängend
$\forall p, q$

nicht wegzusammenhängend

20.5 Satz

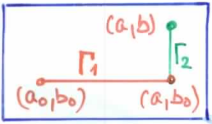
$M \subseteq \mathbb{R}^n$ offen, sternförmig

$F : M \to \mathbb{R}^n$ $\quad F_i$ stetig diffbar $i = 1 - n$

exakt $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ alle i, j

$\Rightarrow$ F hat ein Potential

21.3



$$\varphi(a,b) = \int_{\Gamma_1} F\cdot d\vec{x} + \int_{\Gamma_2} F\cdot d\vec{x}$$

$$= \int_{a_0}^{a} \langle F(t,b_0) \,|\, \vec{e}_1 \rangle \, dt + \int_{b_0}^{b} \langle F(a,t) \,|\, \vec{e}_2 \rangle \, dt$$

$$= \int_{a_0}^{a} F_1(t,b_0)\, dt + \int_{b_0}^{b} F_2(a,t)\, dt$$

$$\frac{\partial \varphi}{\partial x}(a,b) \overset{?}{=} F_1(a,b)$$

Hauptsatz $\Rightarrow$ $\displaystyle \frac{\partial}{\partial x} \int_{a_0}^{a} F_1(t,b_0)\, dt = F_1(a,b_0)$

$$F_1(a,b) - F_1(a,b_0) = \int_{b_0}^{b} \underbrace{\frac{\partial F_1}{\partial y}}(a,t)\, dt$$

$\shortparallel$ exakt

Vertauschung

$$\frac{\partial}{\partial x} \int_{b_0}^{b} F_2(a,t)\, dt = \int_{b_0}^{b} \overbrace{\frac{\partial F_2}{\partial x}}(a,t)\, dt$$

$$\frac{\partial \varphi}{\partial y}(a,b) = \frac{\partial}{\partial y} \int_{b_0}^{b} F_2(a,t)\, dt = F_2(a,b)$$

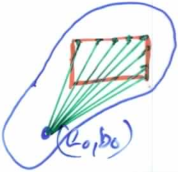
M sternförmig, Zentralpunkt (a_0,b_0)

1. R Rechteck, $\delta(R) \subseteq M$

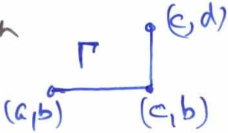
   $\Rightarrow R \subseteq M$

   $\int_\Gamma F\,d\vec{x}$ wegunabhängig für $\Gamma \subseteq R$

   Bew: Potential auf R



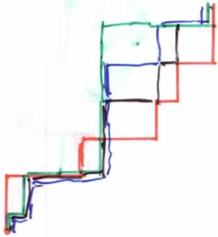
2. Haken



$$\vec{x}(t) = \begin{cases} (t,b) & t\in[a,c] \\ (c,t) & t\in[b,d] \end{cases}$$

3. Hakenweg $\Gamma = \Gamma_1 \oplus \dots \oplus \Gamma_n$ Γ_i: Haken

4. Verfeinerung von Hakenwegen

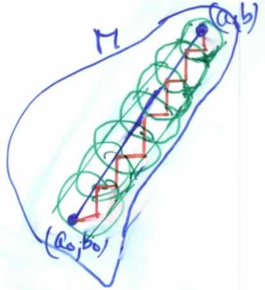


$$\int_\Gamma F\cdot d\vec{x} = \int_\Gamma F\cdot d\vec{x}$$

$$= \int_\Gamma F\cdot d\vec{x} = \int_\Gamma F\cdot d\vec{x}$$

Bew. Wegunabhängigkeit auf Rechtecken

5. Für alle $(a,b) \in M$ gibt es Hakenweg Γ von (a_0,b_0) nach (a,b)



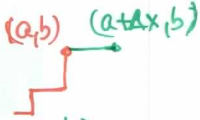
Bew: Kompaktheit der Strecke (a_0,b_0), (a,b)

6. Def $\varphi(a,b) = \int_\Gamma F \cdot d\vec{x}$

unabhängig von Γ Bew 4.

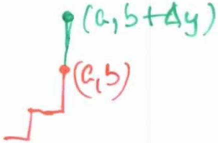
7. $\frac{\partial \varphi}{\partial x}(a,b) = F_1(a,b)$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_a^{a+\Delta x} F_1(t,b)\,dt$$



$$= \frac{\partial}{\partial x} \int F_1(t,b)\,dt\,(a) = F_1(a,b)$$

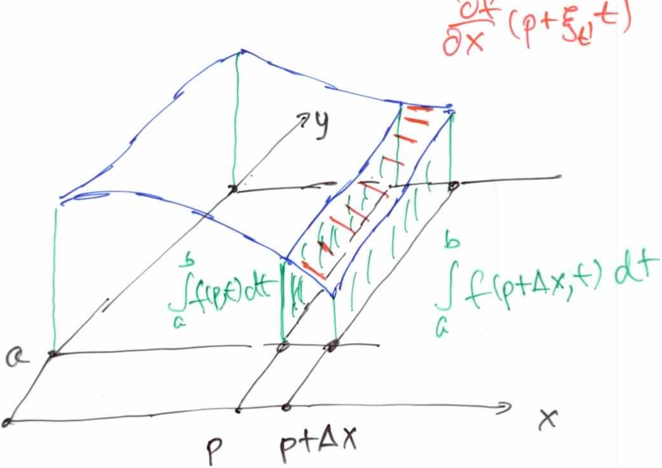
8. $\frac{\partial \varphi}{\partial y}(a,b) = F_2(a,b)$



$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_b^{b+\Delta y} F_2(a,t)\,dt$$

$$= \frac{\partial}{\partial y} \int F_2(a,t)\,dt\,(b) = F_2(a,b)$$

(1)



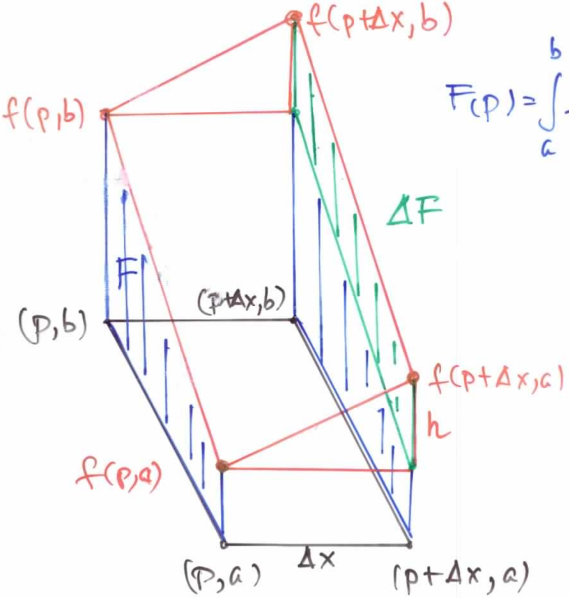
Satz 21.1. Satz $M \subseteq \mathbb{R}^n$ offen

$f: M \times [a,b]$ stetig, $\frac{\partial f}{\partial x_i}$ stet

$$\Longrightarrow \left(\frac{\partial}{\partial x_i} \int_a^b f(x,t)\,dt\right)(p)$$

$$= \int_a^b \frac{\partial f}{\partial x_i}(p,t)\,dt \quad \text{stet Fktnp}$$

$$F(p) = \int_a^b f(p,t)\,dt$$



h = Grundseite
b − a = Höhe

$$\Delta F \approx h(b-a)$$

$$\frac{h}{\Delta x} \approx \frac{\partial f}{\partial x}(p,a)$$

$$\frac{\Delta F}{\Delta x} \approx \frac{\partial f}{\partial x}(p,a)\,(b-a) \approx \int_a^b \frac{\partial f}{\partial x}(p,t)\,dt$$

$$\approx \frac{\partial}{\partial x}\int_a^b f(p,t)\,dt$$

oBdA $n=1$

Schreibweise $\quad x = y \pm \varepsilon \iff y-\varepsilon \leq x \leq y+\varepsilon$

geg. $\varepsilon > 0$ ex. $\delta > 0$: $|\Delta x| < \delta \Rightarrow$

$$\frac{1}{\Delta x}\Big(\underbrace{\int_a^b f(p+\Delta x,t)\,dt - \int_a^b f(p,t)\,dt}_{\int_a^b \big(\underbrace{f(p+\Delta x,t) - f(p,t)}_{\Delta x \left(\underbrace{\frac{\partial f}{\partial x}(p+\xi_{\Delta x},t)}_{\frac{\partial f}{\partial x}(p,t)\pm\varepsilon}\right)}\big)\,dt}\Big)$$

Mittelwert

$\partial f/\partial x$ glm. stetig auf $K \times [a,b]$, $K \subseteq M$ komp.

$$= \frac{1}{\Delta x}\int_a^b \Delta x\left(\frac{\partial f}{\partial x}(p,t) \pm \varepsilon\right) dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}(p,t) \pm \varepsilon\right) dt$$

$$= \int_a^b \frac{\partial f}{\partial x}(p,t)\,dt \pm \underbrace{\varepsilon(b-a)}_{\to 0}$$