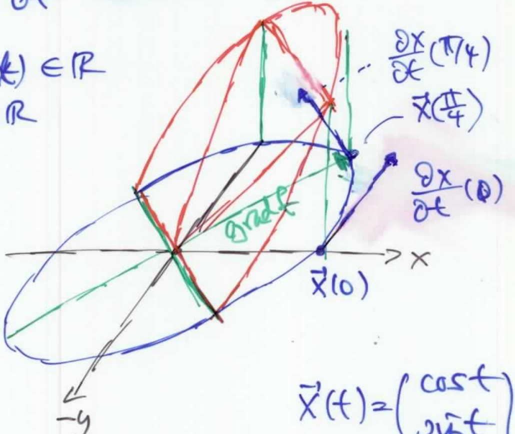
6.3 Kettenregel

$$\frac{\partial f(\vec{x}(t))}{\partial t}(p) = \left\langle \operatorname{grad} f(\vec{x}(p)) \,\middle|\, \frac{\partial \vec{x}}{\partial t}(p) \right\rangle$$

$f(x(t)) \in \mathbb{R}$
$t \in \mathbb{R}$



$$\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

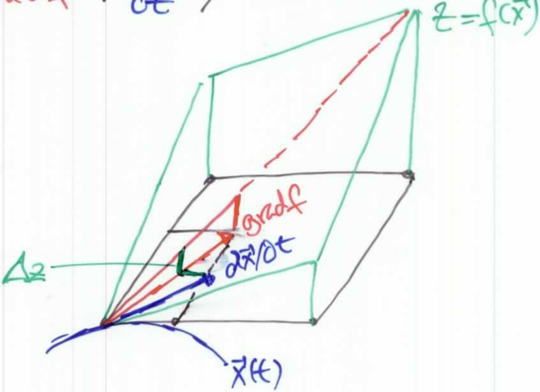
$$z = f(\vec{x}) = \frac{1}{\sqrt{2}}(x+y)$$

$$\frac{\partial \vec{x}}{\partial t} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \qquad df \mathrel{\hat{=}} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\frac{\partial z}{dt} = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \,\middle|\, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle$$

$$= \frac{1}{\sqrt{2}}(-\sin t + \cos t)$$

$$\left\langle \operatorname{grad} f \,\middle|\, \frac{\partial \vec{x}}{\partial t} \right\rangle$$

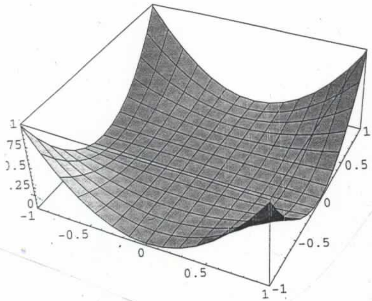


Beweis $\quad f(\vec{x}) = z$

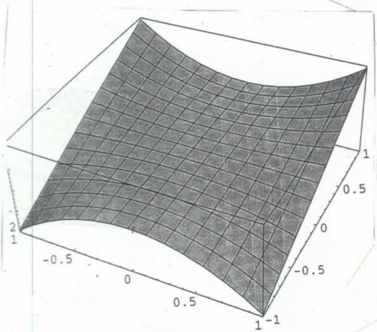
$$\frac{\Delta z}{\Delta t} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \frac{\Delta x_i}{\Delta t} + \frac{R(\Delta \vec{x})\, \|\Delta \vec{x}\|}{\|\Delta \vec{x}\| \;\; \Delta t}$$

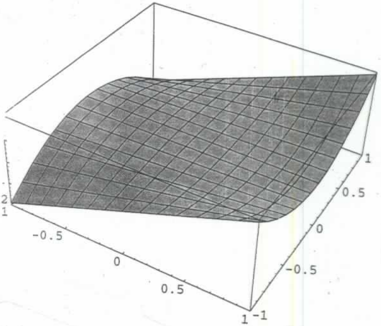
$$\to \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \frac{\partial x_i}{\partial t} + 0$$

$$\frac{R(\Delta \vec{x})}{\|\Delta \vec{x}\|} \to 0 \qquad \frac{\|\Delta \vec{x}\|}{|\Delta t|} \to \left\| \frac{\partial \vec{x}}{\partial t} \right\| < \infty$$



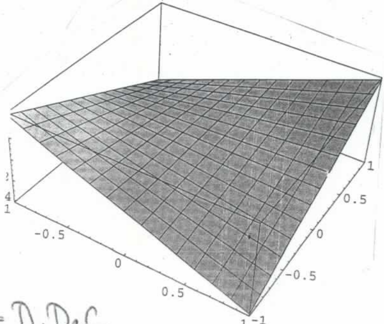
$f(x,y)=x^2+y^2$





$D_2 f$

$D_1 f$



$D_2 D_1 f = D_1 D_2 f$

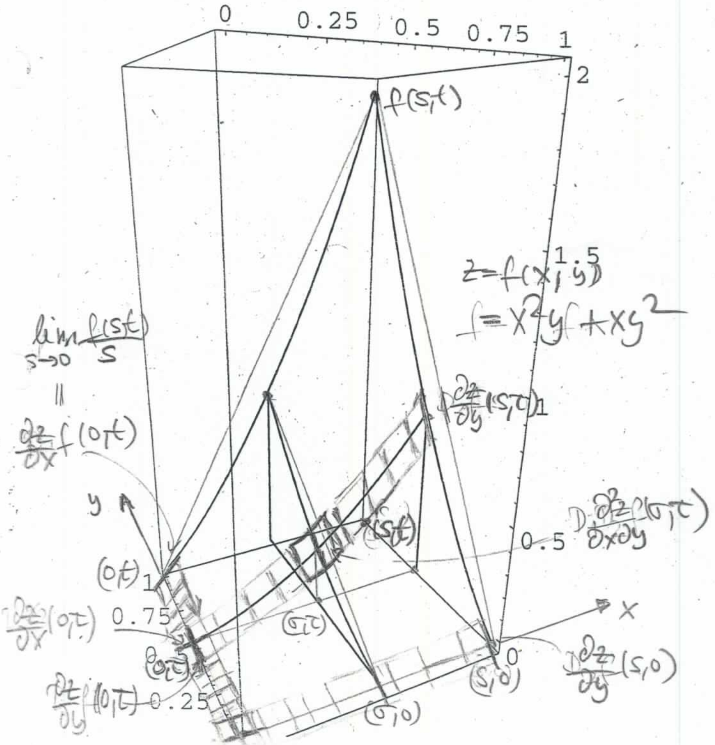
oBdA $n=2$, $p=0$ $\quad f(x,0)=f(0,y)=0$

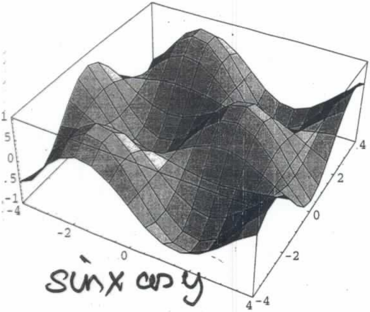
$\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$ ex lokal stetig, $\frac{\partial^2 z}{\partial x \partial y} \to 0 \quad (x,y)\to 0$

$$\frac{1}{t}\,\frac{f(s,t)}{s} = \frac{1}{s}\,\frac{f(s,t)}{t} = \frac{1}{s}\,\frac{\partial z}{\partial y}(s,\tau)$$

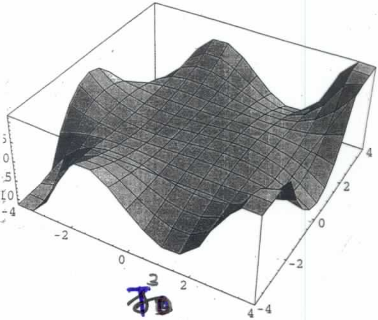
$$= D\,\frac{\partial^2 z}{\partial x \partial y}(\sigma,\tau) \to 0 \quad \text{für } (s,t)\to 0$$

$$\text{also } (\sigma,\tau)\to 0$$

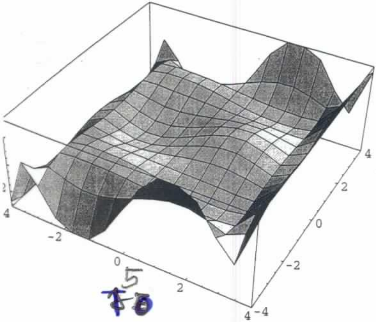




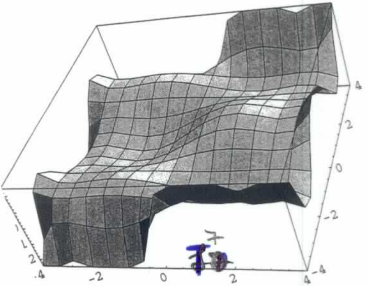
sinx cosy



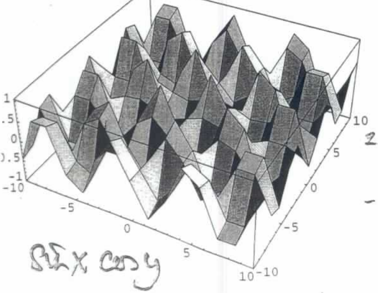
T_{10}^{3}



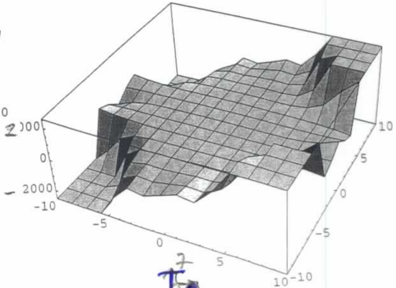
T_{10}^{5}



T_{10}^{7}



sinx cosy



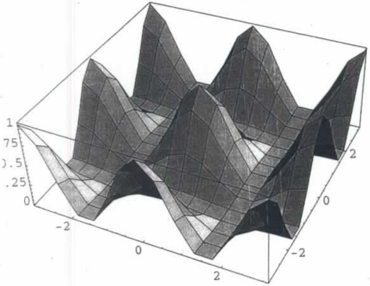
T_{50}^{7}

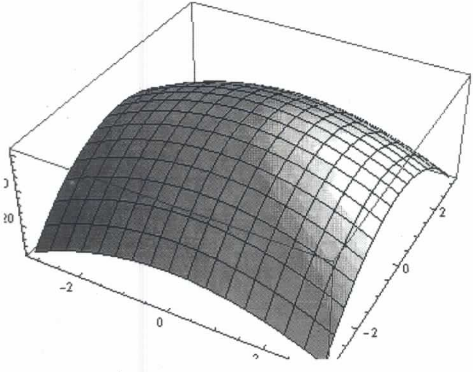
$$f(x,y) = \sin^2 x \cdot \cos^2 y \qquad \vec{p} = \begin{pmatrix} \pi/2 \\ 0 \end{pmatrix}$$

$$\sin(\pi/2 + h_1) = \cos h_1 \underset{0}{\overset{2}{\approx}} 1 - \frac{1}{2} h_1^2$$

$$f(\pi/2 + h_1, h_2) \underset{\vec{p}}{\overset{2}{\approx}} \left(1 - \frac{1}{2} h_1^2\right)^2 \left(1 - \frac{1}{2} h_2^2\right)^2 \underset{\vec{p}}{\overset{2}{\approx}} \left(1 - h_1^2\right) \cdot \left(1 - h_2^2\right)$$

$$\underset{\vec{p}}{\overset{2}{\approx}} 1 - h_1^2 - h_2^2$$





$T^2_{\vec{p}} f$

$$Q(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

$\lambda_1, \lambda_2 > 0$

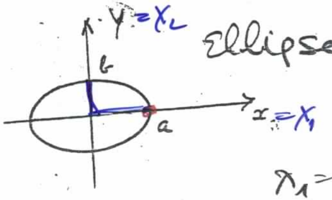
Kennlinie $n=2$

$x_2 = 0$

$\lambda_1 x_1^2 = 1$

$x_1 = \pm \frac{1}{\sqrt{\lambda_1}}$

$a = \frac{1}{\sqrt{\lambda_1}}$



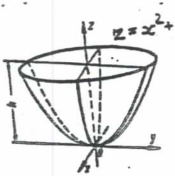
Ellipse

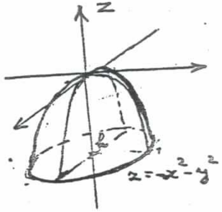
$\lambda_1 = \frac{1}{a^2}$

$$\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1$$

$\lambda_2 = \frac{1}{b^2}$

Graph $n=2$





elliptische Paraboloide

positiv definit

negativ definit

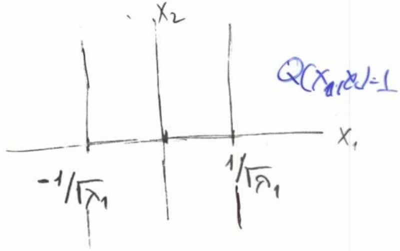
$Q(x_1, x_2) > 0$ $\quad (x_1, x_2) \neq (0,0)$ $\quad Q(x_1, x_2) < 0$

Minimum

Maximum

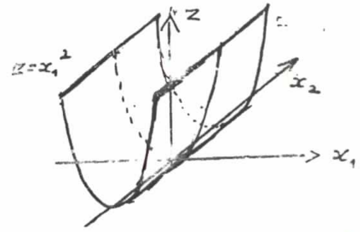
$$Q(x_1, x_2) = \lambda_1 x_1^2 + 0\, x_2^2 \quad = 1$$

$x_1 > 0$



Kennlinie: Geradenpaar

Fläche



positiv semidefinit $Q(x_1,x_2) \geq 0$

parabolischer Zylinder

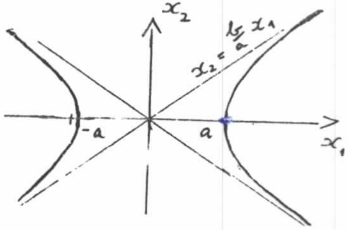
$$Q(x_1,x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2 = 1$$

$\lambda_1 > 0,\ \lambda_2 < 0$

$$a = \frac{1}{\sqrt{\lambda_1}} \quad b = \frac{1}{\sqrt{-\lambda_2}}$$

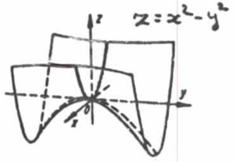
$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1 \approx 0$$

$$\frac{x_2}{x_1} = \pm \frac{b}{a}$$



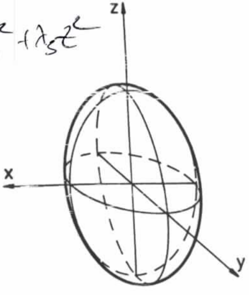
Kennlinie Hyperbel

Fläche:
hyperbolisches
Paraboloid
Sattelfläche
indefinit



$Q(\vec{x}) = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$

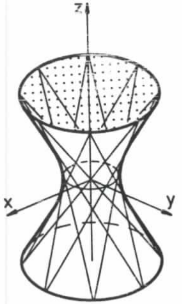
$Q(\vec{x}) = 1$



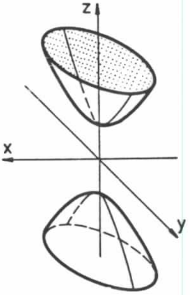
Ellipsoid

$\lambda_i > 0$

positiv definit



Einschaliges Hyperboloid



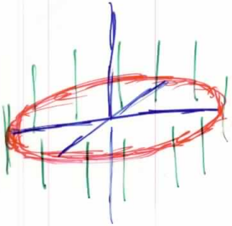
Zweischaliges Hyperboloid

$\lambda_1, \lambda_2 > 0 \ , \ \lambda_3 < 0$

$\lambda_1, \lambda_2 < 0 \ , \ \lambda_3 > 0$

indefinit

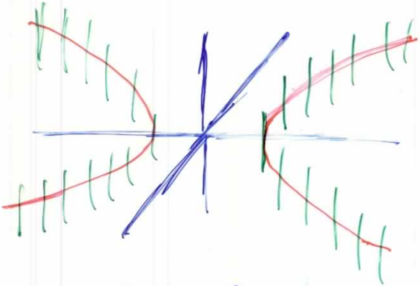
$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 1$$



$\lambda_1 x_1^2 + \lambda_2 x_2^2 + 0 x_3^2 = 1$

$\lambda_1, \lambda_2 > 0$

elliptischer Zylinder



$\lambda_1 x^2 + \lambda_2 x_2^2 + 0 x_3^2 = 1$

$\lambda_1 > 0 > \lambda_2$

hyperbolischer Zylinder