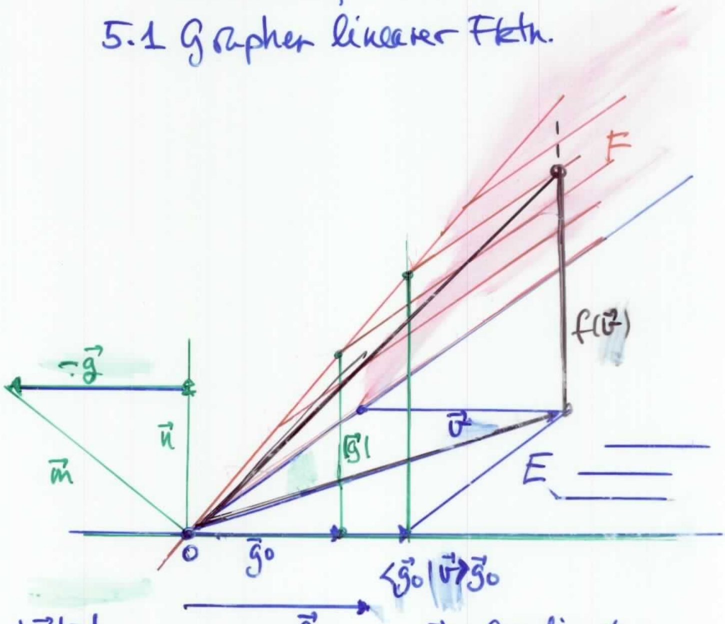
5. Differentiation von Skalarenfeldern

5.1 Graphen linearer Fktn.



$|\vec{g}_0| = 1$
$= |\vec{n}|$
$\vec{n} \perp E$

$\vec{g}$ = Gradient

$f(\vec{x}) = \langle \vec{g} | \vec{v} \rangle$

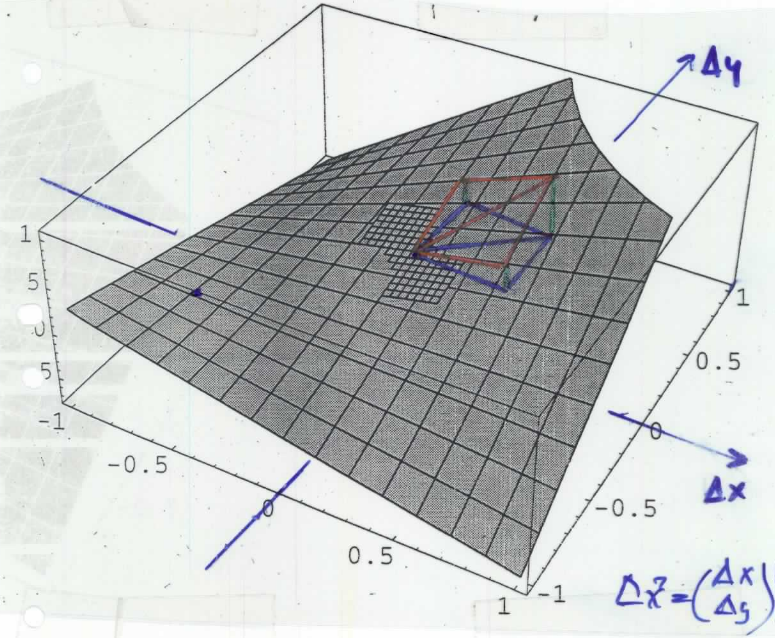
$\vec{g} = \operatorname{grad} f$

$\vec{m} = \vec{n} - \vec{g} \perp F$

$-\vec{g} = \vec{m} - \vec{n}$

$$d_{\vec{v}} = \frac{f(\vec{v})}{|\vec{v}|} = \left\langle \vec{g} \,\middle|\, \frac{\vec{v}}{|\vec{v}|} \right\rangle$$

5.2



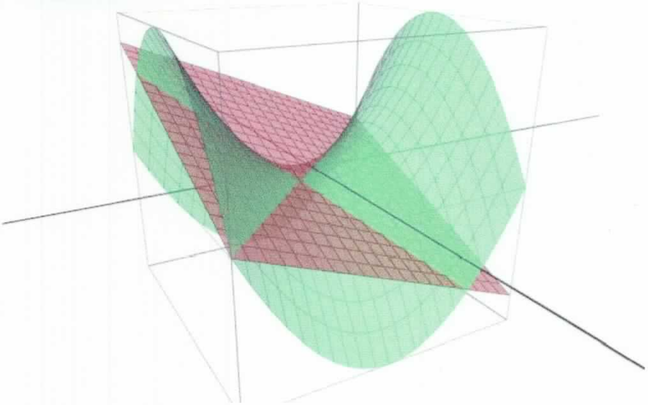
$z = f(x,y) = xy$ $\quad$ $\Delta z = f(\Delta\vec{x}+p) - f(p)$

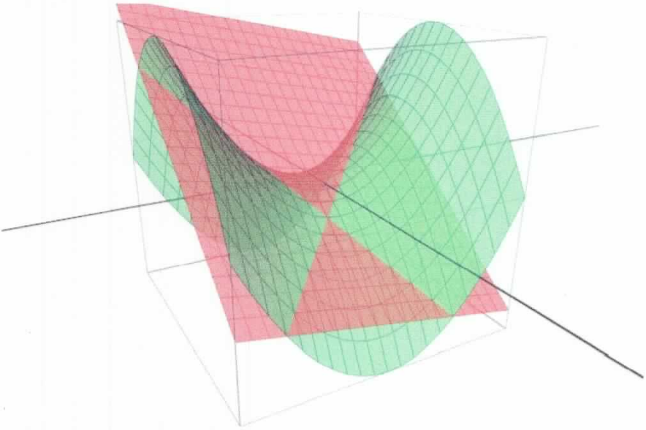
$$\Delta z = (p_1+\Delta x)(p_2+\Delta y) - p_1 p_2$$

$$\Delta z = \underbrace{p_2\Delta x + p_1\Delta y}_{df\binom{p_1}{p_2}\binom{\Delta x}{\Delta y}} + \underbrace{\Delta x\,\Delta y}_{R\binom{\Delta x}{\Delta y}}$$

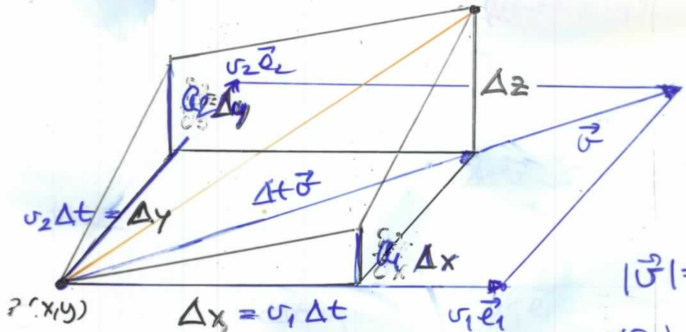
$$\frac{\Delta x\,\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} \to 0$$

5.2





5.3 Richtungsableitung u. Gradient ③



$|\vec{v}|=1$

$\vec{v}=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(p+\Delta t\vec{v})-f(p)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \quad a_1\frac{\Delta x}{\Delta t} + a_2\frac{\Delta y}{\Delta t} + \frac{R(\Delta x,\Delta y)}{\Delta t}$$

$$= \quad a_1 v_1 + a_2 v_2$$

$$= \langle \operatorname{grad} f \,|\, \vec{v} \rangle$$ Ableitung in Richtung $\vec{v}$

$$\operatorname{grad} f = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$ Gradient

$\perp$ Höhenlinie

max in Richtung grad f

5.4

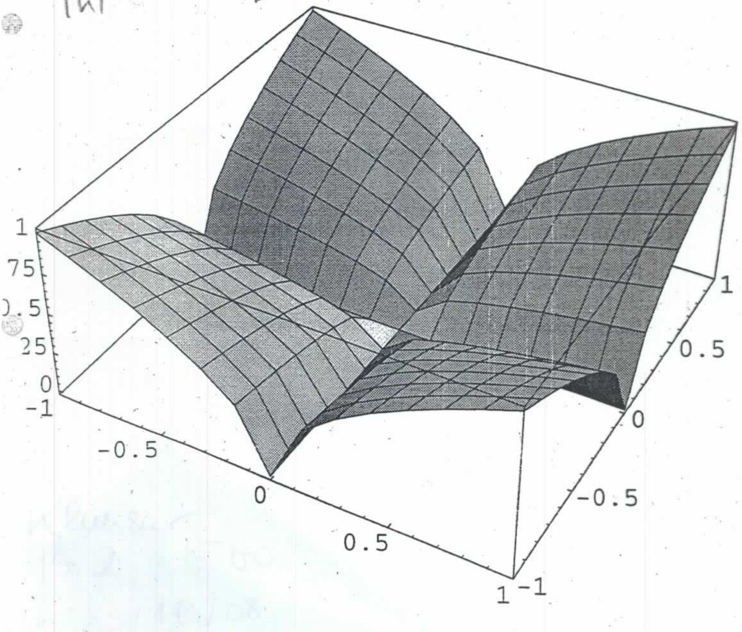
$z = f(x,y) = \sqrt{|xy|}$

$\frac{\partial z}{\partial x}(x,y) = \mp \frac{y}{2\sqrt{|x|}}$ $x \neq 0$ / $\frac{\partial z}{\partial y}(x,y) = \mp \frac{x}{2\sqrt{|y|}}$ $y \neq 0$

$\frac{\partial z}{\partial x}(x,0) = \frac{\partial z}{\partial y}(0,y) = 0$

nicht diffbar an $(0,0)$ - sonst $R(\vec{h}) = f(\vec{h})$

aber $\frac{1}{|\vec{h}|} R(\vec{h}) = \frac{|t|}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}}$ für $\vec{h} = (t,t) \xrightarrow[t \to 0]{} \vec{0}$

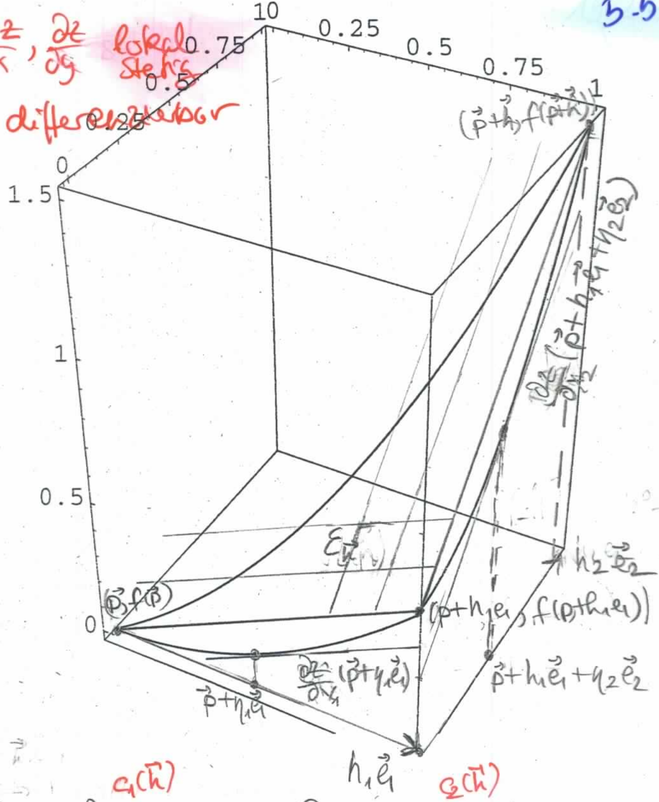


$\vec{p} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{h} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$ $\quad z = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} x_1^2 + x_2^2$

5.5

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ lokal stetig

$\Rightarrow$ differenzierbar



$$R\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \overbrace{\frac{\partial z}{\partial x_1}(\vec{p}+\eta_1\vec{e}_1)}^{c_1(\vec{h})}\Delta x_1 + \overbrace{\frac{\partial z}{\partial x_2}(\vec{p}+\Delta x_1\vec{e}_1+\eta_2\vec{e}_2)}^{c_2(\vec{h})}\Delta x_2$$

$$- \underbrace{\frac{\partial z}{\partial x_1}(\vec{p})}_{a_1}\Delta x_1 - \underbrace{\frac{\partial z}{\partial x_2}(\vec{p})}_{a_2}\Delta x_2 = \left\langle \underbrace{\begin{pmatrix} a_1 - c_1(\vec{h}) \\ a_2 - c_2(\vec{h}) \end{pmatrix}}_{\to \vec{0}} \middle| \vec{h} \right\rangle$$

$$R\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \Big/ \underbrace{\sqrt{\Delta x_1^2 + \Delta x_2^2}}_{\|\vec{h}\|} \longrightarrow 0 \text{ für } \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \to \vec{0}$$

5.5

$$z = f(x,y) = \begin{cases} \sqrt{|xy|}\; e^{-\left(\frac{1}{x}-\frac{1}{y}\right)^2} & xy \neq 0 \\ 0 \end{cases}$$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ ex überall
stetig für $(x,y) \neq (0,0)$

keine Tangentialebene an $(0,0)$

Beachte für $\gamma(t) = (t,t)$ $t \neq 0$

$f(\gamma(t)) = t$ $(f\circ\gamma)'(0) = 1 \neq \underbrace{\sum \frac{\partial f}{\partial x_i}}_{0} \cdot \frac{\partial \gamma_i}{\partial t}$