Aufgabe 1 Let $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined by

$$\varphi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ -x_1 + x_2 + x_4 \\ -2x_1 - x_3 \\ -x_1 - 3x_2 - x_3 + x_4 \end{pmatrix}.$$

Find the matrix of φ with respect to the standard basis of $\mathbb{R}^4$. Compute the rank of φ and determine a basis of the image of φ. Is φ invertible? Determine the dimension of the kernel of φ.

Aufgabe 2 Which of the following 3×3–matrices are invertible?

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

Justify your answers. Compute the inverse matrix in case it exists.

Aufgabe 3

(a) Show that the vectors

$$b_1 := \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \quad b_2 := \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad b_3 := \begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$.

(b) Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by

$$\varphi(x) = \begin{pmatrix} 11x_1 + 12x_2 + 6x_3 \\ -5x_1 - 5x_2 - 3x_3 \\ -3x_1 - 4x_2 \end{pmatrix}.$$

Determine the matrix $[\varphi]_B$ of this endomorphism with respect to the basis $B = (b_1, b_2, b_3)$.

Aufgabe 4 Let $u_1, \ldots, u_m$ be elements of a vector space. Prove that the vectors

$$v_k := \sum_{j=1}^{k} u_j, \quad k = 1, \ldots, m,$$

are linearly dependent if and only if $u_1, \ldots, u_m$ are linearly dependent.

Aufgabe 5 Let $U \subseteq \mathbb{C}^4$ be the linear subspace consisting of all vectors $(x_1, x_2, x_3, x_4)^T \in \mathbb{C}^4$ which fulfill the following equations:

$$\begin{array}{rrrrl} x_1 & +x_2 & -ix_3 & -ix_4 & = 0, \\ ix_1 & & +x_3 & & = 0, \\ & x_2 & & -ix_4 & = 0. \end{array}$$

(a) Find a basis of U.

(b) Let W be the linear subspace of $\mathbb{C}^4$ spanned by the vectors $(1, 0, 0, 0)^T$, $(0, 1, 0, 0)^T$. Prove that $\mathbb{C}^4 = U \oplus W$.

(c) Show that there exists a linear map $\varphi\colon \mathbb{C}^4 \to \mathbb{C}^4$ such that $\varphi(u) = u$ for all $u \in U$ and $\varphi(w) = -w$ for all $w \in W$.

Aufgabe 6

(a) Prove or disprove: There exists a linear map $\varphi\colon \mathbb{C}^6 \to \mathbb{C}^7$ such that $\dim(\ker(\varphi)) = 4$ and $\operatorname{rank}(\varphi) = 3$.

(b) Prove or disprove: There exists an invertible 3×3–matrix A such that $\operatorname{tr}(A) = 0$.

(c) Let $V = M_2(\mathbb{C})$ be the vector space of complex 2×2–matrices. Which of the following maps $V \to \mathbb{C}$ are linear?

$$(a) \quad A \mapsto \det(A), \qquad (b) \quad A \mapsto \operatorname{tr}(A), \qquad (c) \quad A \mapsto \operatorname{rank}(A).$$

Justify your answers.

(d) Let $G = (\mathbb{R}^n, +, 0)$ the additive group of vector addition over $\mathbb{R}^n$. Decide which of the following subsets are subgroups of G.

$$\begin{aligned} A &= \{(x_1, \ldots, x_n)^T \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 1\}, \\ B &= \{(x_1, \ldots, x_n)^T \in \mathbb{R}^n \mid x_1, \ldots, x_n > 0\}, \\ C &= \{(x_1, \ldots, x_n)^T \in \mathbb{R}^n \mid x_1, \ldots, x_n \in \mathbb{Z}\}. \end{aligned}$$

Justify your answers.