Prüfungsdetails

- ca. 3 Wochen vor Semsesterende wird es ein Testats-Übungsblatt geben
- Prüfungsmodalitäten werden genau spezifiziert auf dem Blatt
- Lösungen müssen in den beiden verbleibenden Übungen vorgestellt werden
- Lösungen müssen englische Komentare enthalten
- Lösungen müssen per email an lorenz@mathematik.tu-darmstadt.de
  gesendet werden, und zwar mit dem Subject [IMS2010] und dem Inhalt
  Matrikelnummer1
  Matrikelnummer2
  ...
  und im Anhang: das .mw Maple-file

Sequences, Limits and Series

Computations of limits

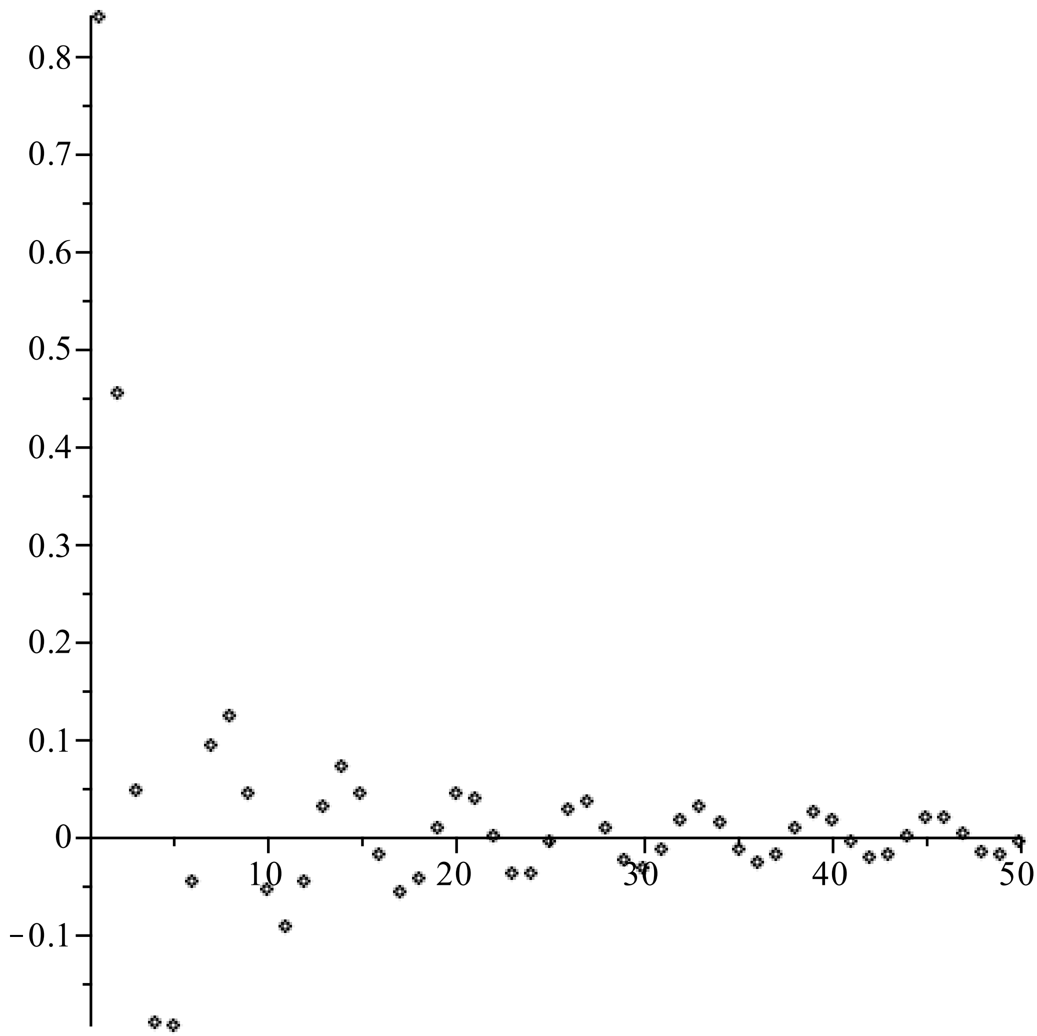
Little dictionary:
limit : Grenzwert
sequence : Folge
series : Reihe

Definition (*sequence*): A **sequence** of real numbers is a mapping from $\mathbb{N} \to \mathbb{R}$.
Example: Let a_n := 1/n, n ⩾1. This gives the sequence (1, 1/2, 1/3, ...)

Definition (*convergence,limit*): Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. A sequence is called convergent towards a $\in\mathbb{R}$, if and only if:
For all $\epsilon > 0$ it exists an $N(\epsilon) \in \mathbb{N}$ such that
$|a_n - a| < \epsilon$ for all $n \geq N(\epsilon)$.
We write $\lim_{n \to \infty} a_n$ = a.

```
> restart;
>
> plots[pointplot]({seq([x, 1/x·sin(x)], x = 1..50)});
```

$$plots[pointplot]\left(\left\{seq\left(\left[x, \frac{1}{x}\cdot\sin(x)\right], x = 1..50\right)\right\}\right);$$



> $limit\left(\frac{\sin(x)}{x}, x = infinity\right)$;

$$0 \tag{1}$$

> $limit\left(\frac{n^2}{n^3+1}, n = \infty\right)$;

$$0 \tag{2}$$

> $limit\left(\frac{\mathrm{Pi}\cdot n^3 + 17\cdot n + n}{n^3+39}, n = \infty\right)$;

$$\pi \tag{3}$$

> $limit\left(\frac{n^k}{n!}, n = infinity\right)$;

$$0 \tag{4}$$

> $limit\left(\frac{n^n}{n!}, n = infinity\right)$;

$$\infty \tag{5}$$

> $limit\left(\frac{n^k}{n!}, n = 0\right);$

$$\lim_{n \to 0} \frac{n^k}{n!} \tag{6}$$

> $limit\left(\frac{n^k}{n!}, n = 0\right)$ assuming $k > 0$;

$$0 \tag{7}$$

> $limit\left(\frac{n^k}{n!}, n = 0\right)$ assuming $k < 0$;

$$\infty \tag{8}$$

Limits of computations:
(there are sequences, the members of which cannot be computed)

Definition (*Turingmachine*): A Turing machine is a formal computation model.

Formally:

* Q is a finite set of states
* Γ is a finite set of the tape alphabet/symbols
* b ∈ Γ is the blank symbol (the only symbol allowed to occur on the tape infinitely often at any step during the computation)
* $\Sigma \subseteq \Sigma \setminus \{b\}$ is the set of input symbols
* $\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L,R,N\}$ is a partial function called the transition function, where L is left shift, R is right shift.
* q0 ∈ Q is the initial state
* F ⊆ Q is the set of final or accepting states.

Example (from Wikipedia):
The 7-tuple for the 3-state busy beaver looks like this (see more about this busy beaver at Turing machine examples):

Q = { A, B, C, HALT }
Γ = { 0, 1 }
b = 0 = "blank"
Σ = { 1 }
δ = see state-table below
q0 = A = initial state
F = the set of final states {HALT}

Initially all tape cells are marked with 0.
State table for 3 state, 2 symbol busy beaver

state	read	write	head	next state
A	0	1	r	B
A	1	1	l	C
B	0	1	l	A

B	1	1	r	B
C	0	1	l	B
C	1	1	r	HALT

We now create Turing machines which have to write as many ones as possible to the tape, without running into an endles loop.

a_n := the number of ones that the best busy beaver with n states can write without ending in a loop.

a_n is not computable for large n.

Known: a_2 = 4, a_3 = 6, a_4 = 13, a_5 >= 4098, a_6 >= 4.6 * 10^1439

Computations of Limits of Functions

Definition (*limits at functions*): Let f : D → ℝ a real valued function on the domain D ⊆ ℝ with a point a ∈ ℝ, such that there exists at least one sequence $(a_n)_{n \in \mathbb{N}}$, $a_n \in$ D with $\lim_{n \to \infty} a_n$ = a.

We write

$$\lim_{x \to a} f(x) = c$$

if and only if it is valid:

$\lim_{n \to \infty} f(a_n)$ = c for all $(a_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} a_n$ = a.

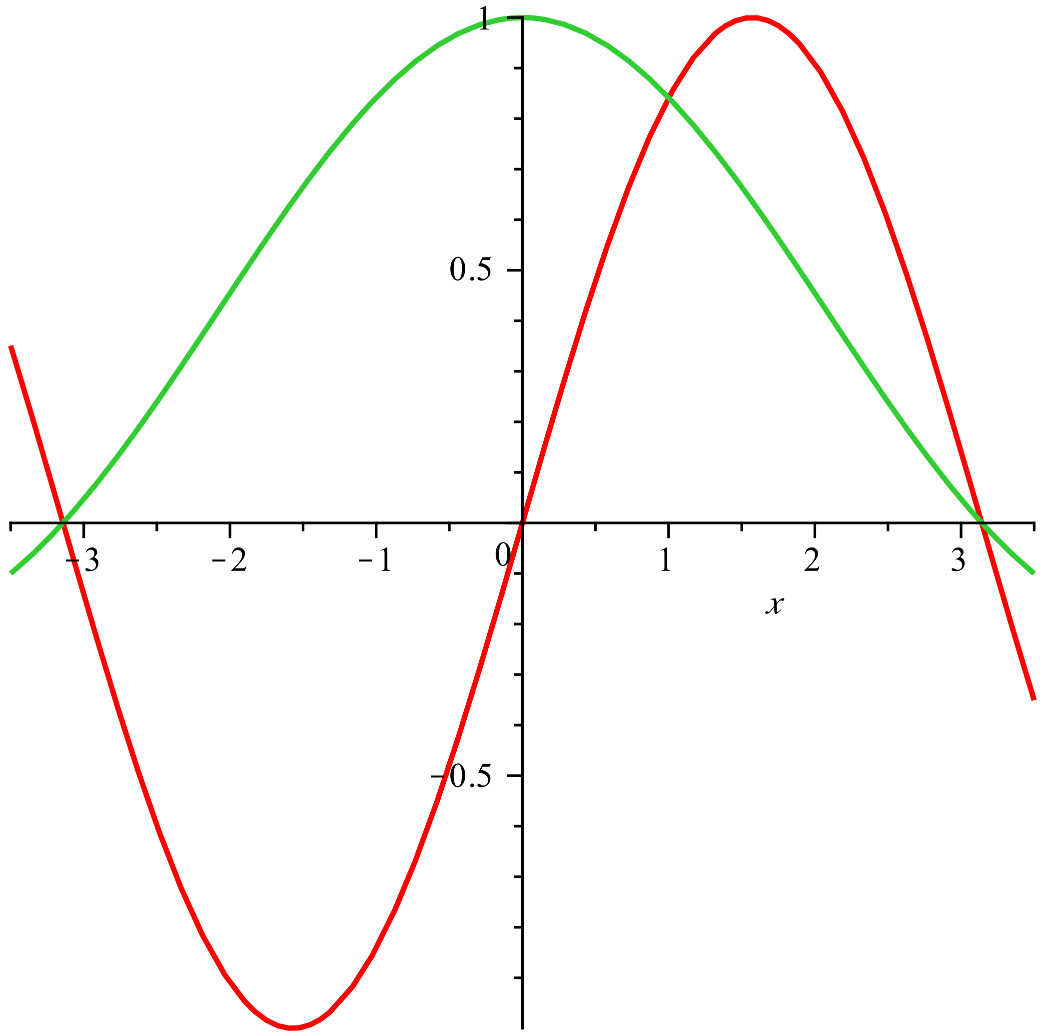
> *limit*(sin(*x*), *x* = 0);

$$0 \tag{9}$$

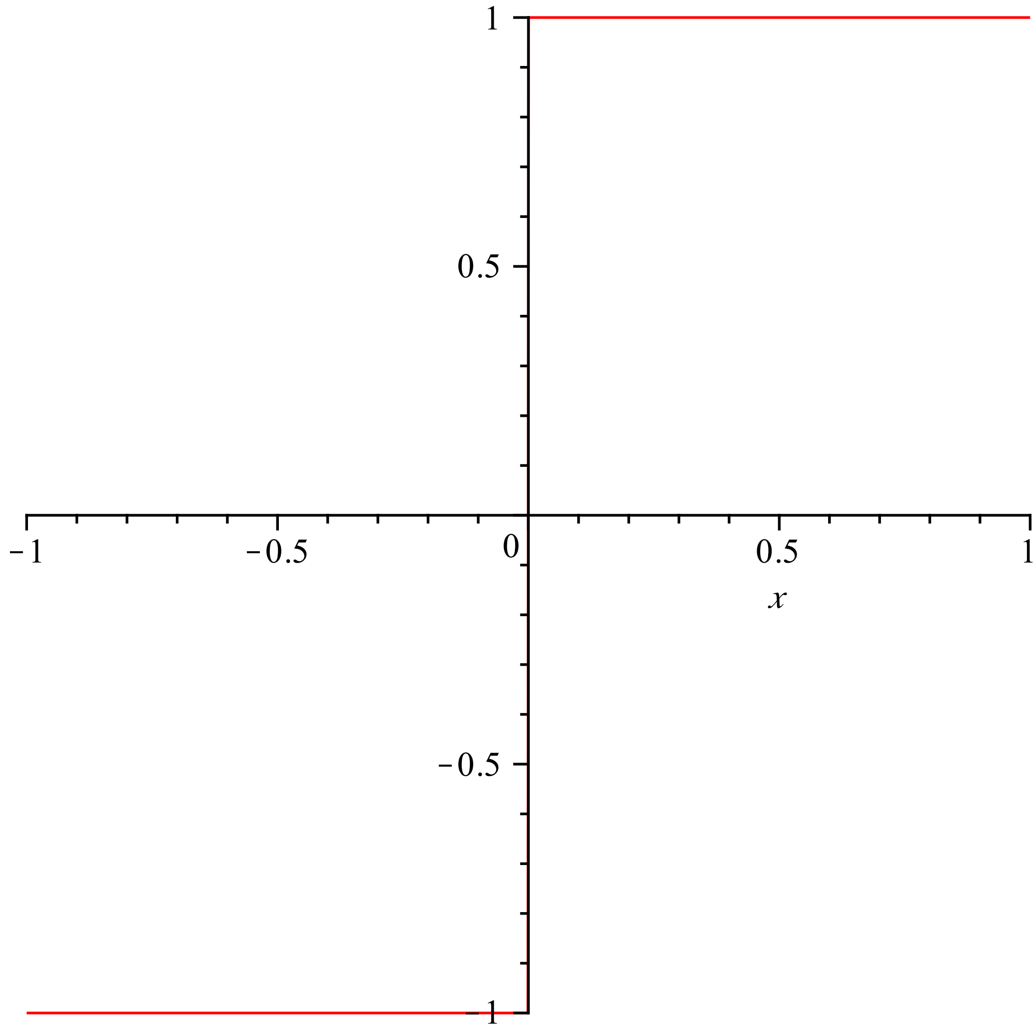
> *limit*$\left(\frac{\sin(x)}{x}, x = 0\right)$;

$$1 \tag{10}$$

> *plot*$\left(\left[\sin(x), \frac{1}{x} \cdot \sin(x)\right], x = -3.5..3.5, thickness = 2\right)$;



> *plot*(signum(*x*), *x*=-1..1);



> $limit(\mathrm{signum}(x), x = 0);$

$$undefined \quad \textbf{(11)}$$

> $limit(\mathrm{signum}(x), x = 0, left);$

$$-1 \quad \textbf{(12)}$$

> $limit(\mathrm{signum}(x), x = 0, right);$

$$1 \quad \textbf{(13)}$$

> $limit(\exp(x), x = infinity);$

$$\infty \quad \textbf{(14)}$$

Computations of Series

Definition (*series*): Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. The sequence $s_n := \sum_{k=0}^{n} a_k$, $n \in \mathbb{N}$ of sums is called **series**, and is described with the help of $\sum_{n=0}^{\infty} a_n$.

Definition (*absolute convergence*): A series $\sum_{n=0}^{\infty} a_n$ is said to **converge absolutely** if the series $\sum_{n=0}^{\infty} |a_n|$ converges, where $|a_n|$ denotes the absolute value of a_n.

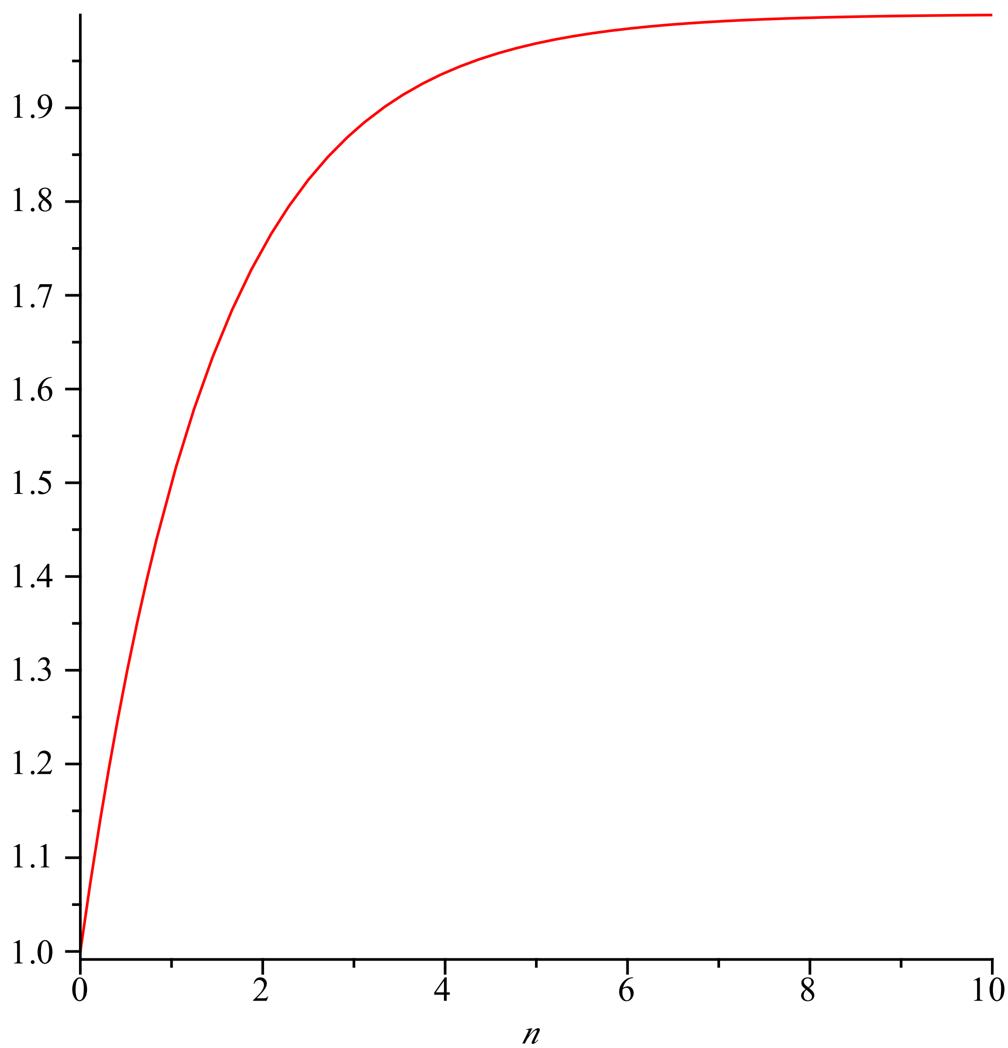
First idea

> *restart*;

> *sum*(*a*[*k*], *k* = 0 ..*infinity*);

$$\sum_{k=0}^{\infty} a_k \qquad \textbf{(15)}$$

> $plot\left(\sum_{i=0}^{n} \left(\frac{1}{2}\right)^i, n = 0..10\right);$



> $sum\left(\left(\frac{1}{2}\right)^n, n = 0..infinity\right);$

$$2 \quad (16)$$

> $limit\left(\sum_{i=0}^{n}\left(\frac{1}{2}\right)^{i}, n=infinity\right);$

$$2 \quad (17)$$

> $\sum_{i=0}^{n}\left(\frac{1}{2}\right)^{i};$

$$-2\left(\frac{1}{2}\right)^{n+1}+2 \quad (18)$$

> $f := \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - \sum_{i=0}^{n}\left(\frac{1}{2}\right)^{i};$

$$f := 0 \quad (19)$$

> $\sum_{i=0}^{\infty} x^{i}$ assuming $x > 1$;

$$\infty \quad (20)$$

> *#computing with series*

> $\sum_{i=0}^{12} x^{i} + \sum_{i=15}^{\infty} x^{i}$ assuming $x < 1$;

$$1+x+x^{2}+x^{3}+x^{4}+x^{5}+x^{6}+x^{7}+x^{8}+x^{9}+x^{10}+x^{11}+x^{12}-\frac{x^{15}}{x-1} \quad (21)$$

> $simplify(\%);$

$$-\frac{1+x^{15}-x^{13}}{x-1} \quad (22)$$

Harmonic Series

> $Harmonic := \sum_{i=1}^{\infty}\frac{1}{i};$

$$Harmonic := \infty \quad (23)$$

> $\sum_{i=1}^{\infty}(-1)^{i}\frac{1}{i};$

$$-\ln(2) \quad (24)$$

> $AlternatingHarmonic := n \rightarrow \frac{(-1)^{n}}{n};$

$$AlternatingHarmonic := n \rightarrow \frac{(-1)^{n}}{n} \quad (25)$$

> *map*(*AlternatingHarmonic*, [*seq*(1 ..10)]);

$$\left[-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, -\frac{1}{7}, \frac{1}{8}, -\frac{1}{9}, \frac{1}{10}\right] \quad \textbf{(26)}$$

> *sum*(*AlternatingHarmonic*(*n*), *n* = 1 ..*infinity*)

$$-\ln(2) \quad \textbf{(27)}$$

The Riemann Series Theorem

Theorem: Let f : ℕ→ℝ be a sequence such that the series $\sum_{k=1}^{\infty} f(k)$ converges but not absolutely.

Then: For each real x there is a bijection (a re-ordering) β:ℕ→ℕ such that $\sum_{k=1}^{\infty} f(\beta(k)) = x$.

We want to construct such a reordering for given f and x. First we need two short functions which will be helpful.

>

Background know-how

Limits of Floating-Point arithmetic in C

```
#include <stdio.h>
int main(void) {
   double x=0.7;
   int i = 0;
   while(i < 10) {
        x = 11.0 * x – 7.0;
        printf(“%d: %.20lf\n”,i,x);
        i=i+1;
   }
}
```

The result of the C-program is rubbish. In the last round it is
y = -1127140547773912.5

Limits of floating-point arithmetic in Maple and of computational speed

> *restart*; $x := \frac{7.0}{10}$;

$$x := 0.7000000000 \quad \textbf{(28)}$$

```
> for i from 1 to 30 do
    x := 11·x - 7;
  end do:
```

> $x;$

$$0.700000000 \quad (29)$$

> $restart, x := \frac{1.0}{3}:$

```
> for i from 1 to 30 do
    x := 3·x - 2/3;
  end do:
```

> $x;$

$$-10294.22328 \quad (30)$$

>

```
> x := 0 : t := time( ) :
  for i from 1 to 5000000 do
    r := rand( ) mod 10;
    for j from 1 to r do
      x := x + 1;
    end do:
  end do:
  x, time( ) − t;
```

$$22492822, 27.480 \quad (31)$$