1.1)

$$sum\left(\left(\frac{1}{2}\right)^n, n = 0..infinity\right)$$

$$2 \tag{1}$$

Geometrically, we can convince ourselves (of course, this is no proof!) that the geometric series converges by considering a rectangle of size 1x2.

For each summand a_n, we mark a part of the rectangle with area $\left(\frac{1}{2}\right)^n$. This way, we always mark half of the area that was still unmarked,
never leaving the area.

1.2)

$$sum\left(\left(\frac{1}{k}\right), k = 1..infinity\right)$$

$$\infty \tag{2}$$

As we see, the harmonic series diverges.

1.3)

$$sum\left(\left(\frac{1}{k^2}\right), k = 1..infinity\right)$$

$$\frac{1}{6}\pi^2 \tag{3}$$

So the sum of the inverses of squares of natural numbers converges.
Two numbers are relatively prime if they don't have any common devisors except 1.

The series of inverses of prime numbers (usually called the Harmonic Series of Primes) diverges, although very slowly.
We could try something like the following:

$$HarmonicPrime := n \rightarrow \frac{1}{ithprime(n)}$$

$$n \rightarrow \frac{1}{ithprime(n)} \tag{4}$$

$$evalf(sum(HarmonicPrime(k), k = 1..361139))$$

$$3.000000031 \tag{5}$$

However, plugging in infinity as upper bound will not yield a result.

1.4)

$sum\left(\frac{(-1)^{k-1}}{k}, k = 1 ..infinity\right)$

$$\ln(2) \tag{6}$$

The alternating harmonic series converges to the natural logarithm of 2.

1.5)

$AlternatingHarmonic := n \rightarrow \frac{(-1)^n}{n};$

$$n \rightarrow \frac{(-1)^n}{n} \tag{7}$$

```
FindNextPositiveStartingIndex := proc( f, k)
local i;
 i := k;
 while f(i) ≤ 0 do
   i := i + 1
  end do;
  return i
end proc;
```

$$\mathbf{proc}(f, k)\ \mathbf{local}\ i;\ i := k;\ \mathbf{while}\ f(i) <= 0\ \mathbf{do}\ i := i + 1\ \mathbf{end\ do};\ \mathbf{return}\ i\ \mathbf{end\ proc} \tag{8}$$

```
FindNextNegativeStartingIndex := proc (f, k)
  local i;
  i := k;
  while 0 ≤ f(i) do
   i := i + 1
  end do;
  return i;
 end proc
```

$$\mathbf{proc}(f, k)\ \mathbf{local}\ i;\ i := k;\ \mathbf{while}\ 0 <= f(i)\ \mathbf{do}\ i := i + 1\ \mathbf{end\ do};\ \mathbf{return}\ i\ \mathbf{end\ proc} \tag{9}$$

$AlternatingHarmonic(2);$

$$\frac{1}{2} \tag{10}$$

```
Riemann := proc (f, x, k)  # here: only for strictly alternating series
  local s, pix, nix, j, p, n, ret;
  s := 0;
  ret := -1;
  pix := FindNextPositiveStartingIndex(f, 1);
  nix := FindNextNegativeStartingIndex(f, 1) ;
  for j from 1 to k do
   if evalf(s) < evalf(x) then
    s := s + f(pix);
    ret := f(pix) ;
    pix := pix + 2;
   else
    s := s + f(nix);
```

```
    ret := f(nix) ;
    nix := nix + 2;
  end if;
 end do;
 return [ret, evalf(s)]
end proc:
```

$seq(Riemann(AlternatingHarmonic, \mathrm{Pi}, i)[1], i = 1000..1020);$

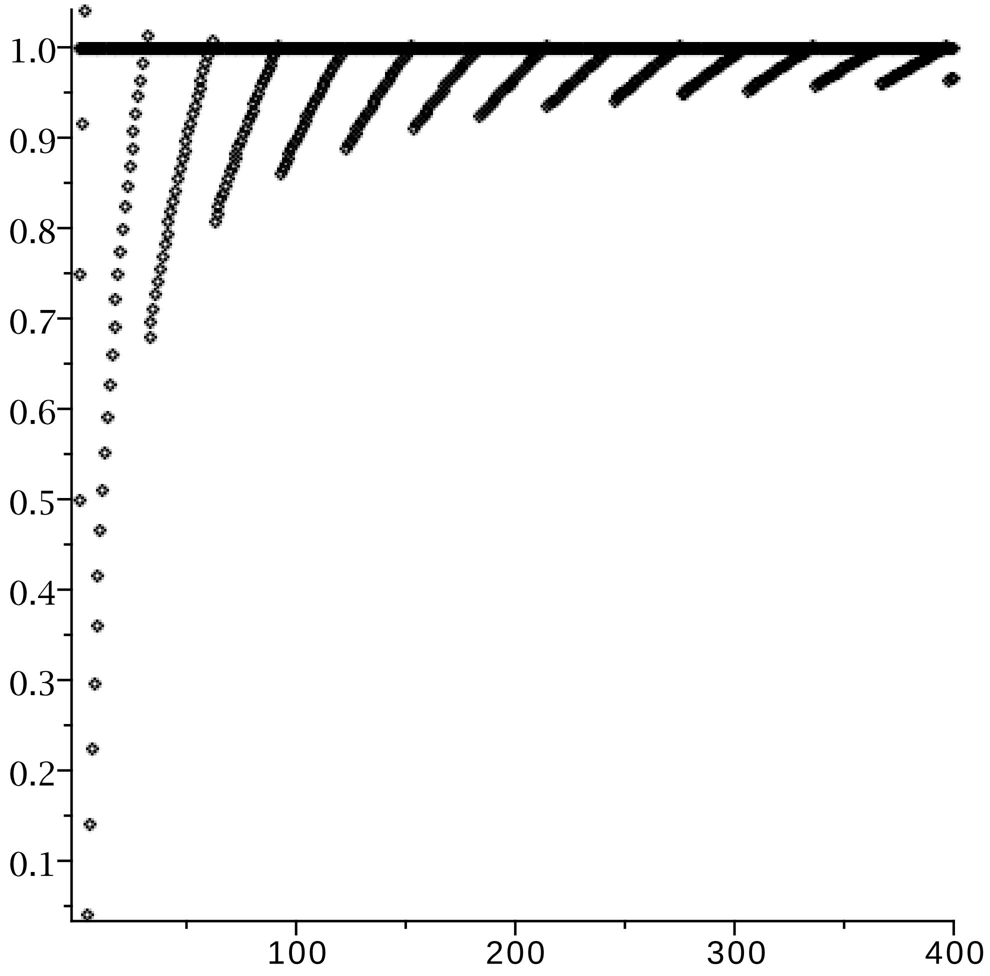
$$\frac{1}{1998}, \frac{1}{2000}, \frac{1}{2002}, \frac{1}{2004}, \frac{1}{2006}, \frac{1}{2008}, \frac{1}{2010}, \frac{1}{2012}, \frac{1}{2014}, \frac{1}{2016}, \frac{1}{2018}, \frac{1}{2020}, \frac{1}{2022}, \frac{1}{2024}, \frac{1}{2026}, \frac{1}{2028}, \frac{1}{2030}, \frac{1}{2032}, \frac{1}{2034}, \frac{1}{2036}, \frac{1}{2038} \tag{11}$$

$seq\left(Riemann\left(AlternatingHarmonic, \frac{100}{1001}, i\right)[2], i = 1000..1020\right);$

$$0.09996687477, 0.09703432345, 0.09763600817, 0.09823696971, 0.09883720981, 0.09943673019, 0.1000355326, 0.09712008069, 0.09771816681, 0.09831553838, 0.09891219709, 0.09950814465, 0.1001033827, 0.09720483202, 0.09779936234, 0.09839318657, 0.09898630638, 0.09957872344, 0.1001704394, 0.09728859504, 0.09787961158 \tag{12}$$

$rseq := seq\left(\left[i, Riemann\left(AlternatingHarmonic, \frac{1000}{1001}, i\right)[2]\right], i = 1..400\right):$

$plots[pointplot]\left(\left\{seq\left(\left[x, \frac{1000}{1001}\right], x = 1..400\right), rseq\right\}\right);$



2.)

```
SeriesProduct := proc(a, b, k)
  local i, j, s;
  s := 0;
  for i from 0 to k do
    for j from 0 to i do
      s := s + a(j) · b(i − j);
    end do;
  end do;
  return s;
 end proc;
```

```
proc(a, b, k)
    local i, j, s;
    s := 0;
    for i from 0 to k do for j from 0 to i do s := s + a(j) * b(i − j) end do end do;
    return s
end proc
```
(13)

$a := n \rightarrow \frac{1}{(n+1)^3}$

$$n \rightarrow \frac{1}{(n+1)^3} \tag{14}$$

$b := n \rightarrow \frac{1}{(n+1)^2}$

$$n \rightarrow \frac{1}{(n+1)^2} \tag{15}$$

$evalf(SeriesProduct(a, b, 1000))$

$$1.976102822 \tag{16}$$

$seq(evalf(SeriesProduct(a, b, i)), i = 1000..1010);$

$$1.976102822, 1.976104021, 1.976105219, 1.976106414, 1.976107607, 1.976108797, 1.976109985, 1.976111170, 1.976112354, 1.976113534, 1.976114713 \tag{17}$$

$evalf(sum(a(x), x = 0..infinity) \cdot sum(b(x), x = 0..infinity))$

$$1.977304350 \tag{18}$$