Introduction to Mathematical Software Exercise 7

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Problem 1 Infinite Series

Problem 1.1 The geometric series

Consider the series

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots,$$

which is the sum of infinitely many numbers. Does this correspond to a finite number, i.e. does this series converge? If you don't know already, make a guess and check it using Maple. Can you find a geometric "proof" why this series converges or diverges?

Problem 1.2 The harmonic series

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

Check whether this series converges. Can you explain why it does or doesn't converge?

Problem 1.3 Other series

Apparently, the numbers that are summed up have to decrease fast enough for a series to converge. But how fast is fast enough?
Consider the sum of inverses of the squares of natural numbers,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \\ &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots\end{aligned}$$

Check whether this series converges. This is the so called "Basel problem" in number theory, posed by Pietro Mengoli in 1644 and solved by Euler in 1735. Its states the inverse of the probability of two natural numbers being relatively prime (what does that mean?)
Finally, consider the sum of (inverses of) prime numbers

$$\sum_{p \text{ is prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \ldots$$

Can you use Maple to figure out whether this series converges?

Problem 1.4 The alternating harmonic series

We have seen that the summands have to decrease fast enough for the series to converge. Another possibility are changing signs:
Consider the “alternating harmonic series”

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \dots$$

Does this series converge or diverge?

Problem 1.5 Riemann series theorem

We now consider a very remarkable property of infinite series known as the Riemann series theorem, which is also called the Riemann rearrangement theorem:

Theorem (Riemann series theorem)
Let (a_n) be a conditionally convergent (i.e. convergent but not absolutely convergent) series and let M be a real number. Then, there exists a permutation $\sigma(n)$ of the natural numbers, such that

$$\sum_{k=1}^{\infty} a_{\sigma(n)} = M.$$

This means in particular that the summation of infinitely many numbers is **not commutative!**
Devise an algorithm that, for a given conditionally convergent series (a_n) and a real number M, creates a permutation of the summands such that the rearranged series converges to M.
Implement your algorithm as a Maple procedure.

Problem 2 Cauchy-Product of series

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series. Find an algorithm for the computation of c_n with $(\sum_{n=0}^{\infty} c_n) = (\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n)$, up to a certain index k.
In detail, for $k \in \mathbb{N}$, compute

$$\sum_{n=0}^{k} c_n,$$

with

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

> A mathematician organizes a raffle in which the prize is an infinite amount of money paid over an infinite amount of time. Of course, with the promise of such a prize, his tickets sell like hot cake.
> When the winning ticket is drawn, and the jubilant winner comes to claim his prize, the mathematician explains the mode of payment: "1 dollar now, 1/2 dollar next week, 1/3 dollar the week after that..."