Lecture 9 — Integration

In antiquity, Archimede determined the volume of special bodies such as the cone, sphere, and cylinder. To calculate areas or volumes in general is the main task of integration. The first attempt for a systematic treatment of integration goes back to Cavalieri in the 17th century.

The integral of a function in one variable is the oriented area bounded by the graph. Two questions arise:

- For which functions can we declare the integral?
- How do we compute integrals?

The answer to the second question will be deferred until Section 9.2: The Fundamental Theorem of Calculus will turn out to be crucial.

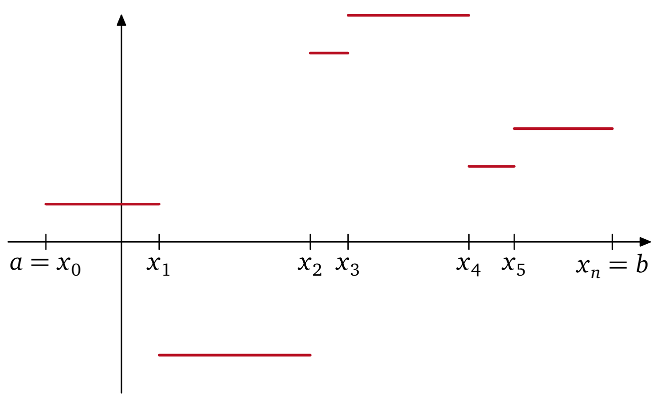
The first question has less practical impact; in fact, all functions occurring in daily life are integrable. It is, however, an interesting mathematical problem. Still, we will in this course mainly restrict our attention to continuous functions, which are always integrable.

9.1 Step functions

Before coming to continuous functions, which we are mainly interested in, we will start with another interesting and easy to handle class of functions.

Definition 9.1.1. A function $\phi\colon [a,b] \to \mathbb{R}$ is a *step function [Treppenfunktion] if there is a partition [Zerlegung]* $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ *of* $[a,b]$ *such that* ϕ *is constant on each interval* (x_{k-1}, x_k) *for* $k = 1, \ldots, n$. *Note that the number* n *of steps is finite and we do not constrain the values* $\phi(x_k)$.

An example of a step function:



Let us denote the set of step functions on $[a,b]$ by $S[a,b]$. The sole point of introducing these functions is that their integrals are obvious since we know how to compute the area of a rectangle:

Definition 9.1.2 (Integral of step functions)**.** Let $\phi \in S[a,b]$ with $\phi(x) = c_k$ on (x_{k-1}, x_k) for $k = 1, \dots, n$. Then we set

$$\int_a^b \phi(x)\,dx := \sum_{k=1}^{n} c_k(x_k - x_{k-1}).$$

We also admit $a = b$, in which case the sum is empty and $\int_a^a \phi(x)\,dx = 0$.

See Figure 9.1 for an illustration.

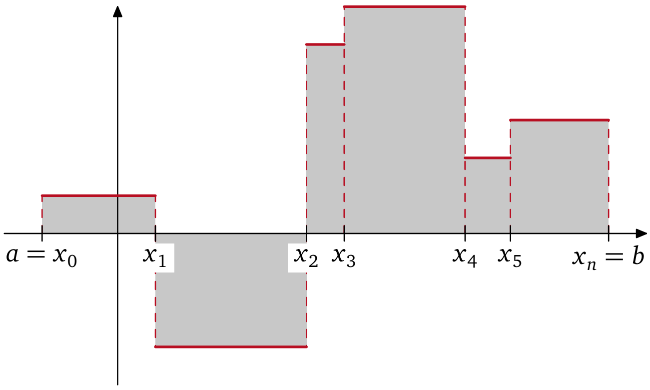


Figure 9.1: The integral over a step function.

Remark 9.1.3. The same step function can be described with respect to many different partitions, for instance, we can always include additional support points into a given partition. Then $\int_a^b \phi(x)\,dx$ remains invariant:

- For just one additional support point, this is seen as follows: If $\phi(x) := c$ on $[a,b]$ and $\xi \in (a,b)$, then

$$c(b-a) = c(\xi - a) + c(b - \xi), \tag{9.1}$$

  since rectangle areas add.

- For the general case, if X is a partition $a = x_0 < x_1 < \ldots < x_i = b$ and Y is $a = y_0 < y_1 < \ldots < y_j = b$ then their union forms a partition Z of form $a = z_0 < z_1 < \ldots < z_k = b$, having $k \le i + j$ points. Appealing to Equation (9.1) above, we see that the sums with respect to X and Z are equal, and so are the sums with respect to Y and Z. Consequently, the sums for X and Y are also equal, which means the integral is well-defined.

9.2 The Fundamental Theorem of Calculus

Definition 9.2.1. A differentiable function $F : [a,b] \to \mathbb{R}$ is called a *primitive* or *antiderivative [Stammfunktion] of* $f : [a,b] \to \mathbb{R}$ *if* $F' = f$.

Example 9.2.2.

(i) For $f(x) = x^2$ the function $F(x) = \frac{1}{3}x^3$ is a primitive.

(ii) For $f(x) = e^x$ the function $F(x) = e^x$ is a primitive.

Often, the explicit form of a primitive can only be guessed. Nevertheless it always exists if f is continuous:

Theorem 9.2.3 (*Fundamental Theorem of Calculus [Fundamentalsatz der Differential- und Integralrechnung]*). *Suppose a continuous function $f\colon [a,b] \to \mathbb{R}$ has a primitive $F\colon [a,b] \to \mathbb{R}$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Let us rephrase the statement, which presents perhaps the most important fact of calculus. The equation $\left(\int_a^x f(t)\,dt\right)' = f(x)$ means that indefinite integration and differentiation are inverse operations, cancelling one another. This is not at all clear from the definition of integral and derivative!

For f constant, *i.e.*, $f(x) = c$, this is immediate to see: If we denote $I(x) := \int_a^x f(x)\,dx$ then $I(x) = (x-a)c$ and so $I'(x) = c$.

Obviously, if F is a primitive of f, then so is $F+c$ for c constant. Conversely, any two primitives $F, G\colon [a,b] \to \mathbb{R}$ of the same function f satisfy

$$(F-G)' = F' - G' = f - f = 0;$$

This implies that $F - G$ is constant. That is, a primitive of f is well-defined up to a constant. Making use of this property we see that the integral of f can be computed using any of its primitives F.

The Fundamental Theorem allows us to integrate most functions introduced so far. It will be convenient to write $F(x)\Big|_a^b := F(b) - F(a)$.

Example 9.2.4.

(i) From the examples for differentiation, the following is immediate:

$$\int_a^b x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_a^b$$

Thanks to the linearity of the integral (*i.e.*, $\int (f+g)\,dx = \int f\,dx + \int g\,dx$) this formula suffices to integrate polynomials.

(ii)

$$\int_a^b e^x\,dx = e^x\Big|_a^b$$

(iii)

$$\int_a^b \cos x\,dx = \sin x\Big|_a^b, \qquad \int_a^b \sin x\,dx = -\cos x\Big|_a^b.$$

(iv) Moreover,

$$\int_a^b \frac{1}{1+x^2}\,dx = \arctan x\Big|_a^b,$$

(v) and, provided $[a,b]$ does not contain a zero of cos,

$$\int_a^b \frac{1}{\cos^2 x}\,dx = \tan x\Big|_a^b.$$

9.3 Rules for integration

Each law of differentiation yields a law for integration, via the Fundamental Theorem.

Let us call a function *continuously differentiable [stetig differenzierbar] if its derivative is continuous.*

We consider the product rule first.

Theorem 9.3.1 (*Integration by parts [partielle Integration]*). *If $f,g\colon [a,b]\to\mathbb{R}$ are continuously differentiable then*

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx.$$

Note that the two integrals on the right hand side exist in view of our assumptions on f,g.

Proof. The function $h := fg$ can be differentiated using the product law: $h' = f'g + fg'$. In particular, h' is continuous, and so

$$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = \int_a^b h'(x)\,dx \overset{\text{Fund'l Thm.}}{=} h(x)\Big|_a^b = f(x)g(x)\Big|_a^b. \qquad \square$$

Example 9.3.2.

$$\int_a^b \cos(x)x\,dx = \sin(x)x\Big|_a^b - \int_a^b \sin(x)\cdot 1\,dx = \sin(x)x\Big|_a^b + \cos(x)\Big|_a^b$$

We now discuss the chain rule. Let us first introduce some more notation.

Suppose $F, f = F'\colon [a,b]\to\mathbb{R}$ and $x,y\in[a,b]$. Then the Fundamental Theorem gives

$$F(y) - F(x) = \int_x^y f(t)\,dt.$$

The same formula will hold for $x > y$ as well, provided we set

$$\int_x^y f(t)\,dt := -\int_y^x f(t)\,dt.$$

Theorem 9.3.3 (*Substitution*)**.** *Let $f : [\alpha, \beta] \to \mathbb{R}$ be continuous and $\phi : [a, b] \to [\alpha, \beta]$ be continuously differentiable. Then*

$$\int_a^b f\left(\phi(t)\right)\phi'(t)\, dt = \int_{\phi(a)}^{\phi(b)} f(x)\, dx. \tag{9.2}$$

Proof. Let $F : [\alpha, \beta] \to \mathbb{R}$ be a primitive of f. According to the chain rule,

$$(F \circ \phi)'(t) = F'\left(\phi(t)\right)\phi'(t) = f\left(\phi(t)\right)\phi'(t),$$

and so (9.2) follows from

$$\int_a^b f\left(\phi(t)\right)\phi'(t)\, dt \overset{\text{Fund'l Thm.}}{=} (F \circ \phi)(t)\Big|_a^b = F\left(\phi(b)\right) - F\left(\phi(a)\right) \overset{\text{Fund'l Thm.}}{=} \int_{\phi(a)}^{\phi(b)} f(x)\, dx. \qquad \square$$

Example 9.3.4.

(i) Integration is invariant under translation in the domain: For $c \in \mathbb{R}$,

$$\int_a^b f(\underbrace{t+c}_{\phi(t)})\, dt \overset{(9.2)}{=} \int_{\phi(a)=a+c}^{\phi(b)=b+c} f(x)\, dx \qquad (\phi'(t) = 1).$$

(ii) For $c \in \mathbb{R}$ and $\phi(t) := ct$ we have

$$\int_a^b f(ct)c\, dt \overset{(9.2)}{=} \int_{ca}^{cb} f(x)\, dx \quad \overset{c\neq 0}{\Longrightarrow} \quad \int_a^b f(ct)\, dt = \frac{1}{c}\int_{ca}^{cb} f(x)\, dx.$$

(iii) Let us now discuss a classical problem: the area of the unit disk. The area of the upper half disk is the integral $\int_{-1}^1 \sqrt{1-x^2}\, dx$. We want to substitute x by $\phi(t) := \sin t$ in order to take advantage of the identity $\sin^2 t + \cos^2 t = 1$. Note that $\phi : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is continuously differentiable and invertible. Substitution gives

$$\int_{-1}^1 \sqrt{1-x^2}\, dx = \int_{\phi^{-1}(-1)}^{\phi^{-1}(1)} \underbrace{\sqrt{1-\sin^2 t}}_{\sqrt{\cos^2 t}}\, \underbrace{(\sin t)'}_{\cos t}\, dt = \int_{-\pi/2}^{\pi/2} \cos^2 t\, dt \overset{Ex.}{=} \frac{\pi}{2}.$$

Here, we used the fact that $\cos t \geq 0$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus the unit disk has area π.