Lecture 8 — Differentiable Functions

Let $U \subseteq \mathbb{R}$ and $f : U \to \mathbb{R}$.

Assume that we want to construct the *tangent [Tangente]* t *to the graph of* f *at a fixed point* $x_0 \in U$. *Since* t *goes through the point* $(x_0, f(x_0))$, *it suffices to determine the slope [Steigung] of* t.

To this end, we first draw a line l_x *through the points* $(x_0, f(x_0))$ *and* $(x, f(x))$. *This line intersects the graph of* f *and is not yet the required tangent. But if we now move the point* x *towards* x_0 *on the* x*-axis, the line* l_x *gets closer and closer to the tangent* t. *In the limit* $x \to x_0$ *(if it exists), the line* l_x *and the tangent* t *coincide. This process is illustrated in Figure 8.1*

Now have a closer look at the slope of l_x. *It is*

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

Since l_x *tends to* t *as* x *tends to* x_0, *the slope of* t *is the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

8.1 Definition of differentiability

Definition 8.1.1. Let $f : (a, b) \to \mathbb{R}$ be a function and $x_0 \in (a, b)$. Then f is called *differentiable [differenzierbar] in* x_0 *if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and coincides for all sequences $x_n \to x_0$. *The derivative [Ableitung] of* f *in* x_0 *is denoted by* $f'(x_0)$. *(Read:* f *prime of* x_0.*)*

If f *is differentiable in each point of* (a, b) *then it is called differentiable on* (a, b). *In this case,* f' *is a function* $(a, b) \to \mathbb{R}$.

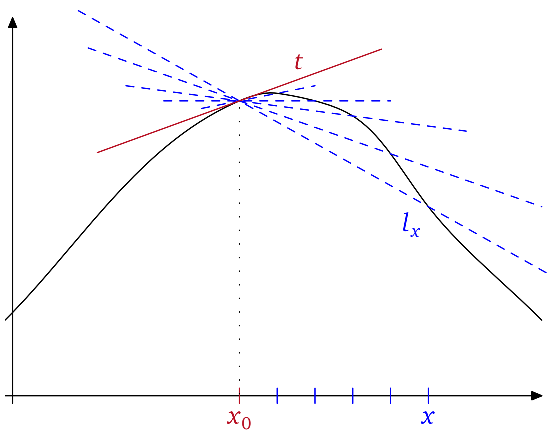

Figure 8.1: The tangent t in a point x_0 can be constructed as the limit of a sequence of secants.

(The last step is more abstract than it seems. It takes us in one stride from a single value $f'(x_0)$ to a function f')

Theorem 8.1.2. *If $f:(a,b)\to\mathbb{R}$ is differentiable in x_0 then f is also continuous in x_0.*

Proof. Let $x_0 \in (a,b)$. Then the limit

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

exists since f is differentiable in x_0. As x goes to x_0, the denominator converges to zero. Hence the limit can only exist if the numerator also converges to zero. If the numerator converges to zero then $f(x)$ converges to $f(x_0)$. In other words,

$$\lim_{x\to x_0} f(x) = f(x_0).$$

This is our definition of continuity. □

On the other hand, continuity of f does not imply differentiability as the following example will show:

Example 8.1.3. Consider the function

$$\begin{aligned} f:\mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto |x|. \end{aligned}$$

We look at $x_0 = 0$ and show that f is continous in x_0. We choose a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ which approaches 0 from above (we denote this by $x_n \searrow 0$). Then this yields

$$\lim_{x_n\searrow 0} f(x_n) = \lim_{x_n\searrow 0}|x_n| \overset{x_n>0}{=} \lim_{x_n\searrow 0} x_n = 0.$$

If we consider a sequence x_n which approaches 0 from below (we denote this by $x_n \nearrow 0$), then we get

$$\lim_{x_n\nearrow 0} f(x_n) = \lim_{x_n\nearrow 0}|x_n| \overset{x_n<0}{=} \lim_{x_n\nearrow 0} -x_n = 0.$$

This shows that f is continous in x_0.

Now let us check whether f is differentiable. Again we choose a sequence $x_n \searrow 0$. This yields

$$\lim_{x_n\searrow 0}\frac{f(x_n)-f(0)}{x_n-0} = \lim_{x_n\searrow 0}\frac{f(x_n)}{x_n} = \lim_{x_n\searrow 0}\frac{|x_n|}{x_n} \overset{x_n>0}{=} \lim_{x_n\searrow 0}\frac{x_n}{x_n} = 1.$$

On the other hand, if we have a sequence $x_n \nearrow 0$, we get

$$\lim_{x_n\nearrow 0}\frac{f(x_n)-f(0)}{x_n-0} = \lim_{x_n\nearrow 0}\frac{f(x_n)}{x_n} = \lim_{x_n\nearrow 0}\frac{|x_n|}{x_n} \overset{x_n<0}{=} \lim_{x_n\nearrow 0}\frac{-x_n}{x_n} = -1.$$

We see that the limits do not coincide, which means that f is not differentiable in $x_0 = 0$.

While a differentiable function is continous, the derivative of a continous function need not be continous.

Another way to write the definition

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

of the derivative of f at x_0 is to write the sequence x_n with limit x_0 as $x_0 + h$ and look at the limit $h \to 0$. Then we get

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Now we look at a few examples and determine some derivatives:

Example 8.1.4.

(i) $f : \mathbb{R} \to \mathbb{R} : x \mapsto c \cdot x$ with $c \in \mathbb{R}$.

$$f'(x_0) = \lim_{x_n \to x_0} \frac{cx_n - cx_0}{x_n - x_0} = \lim_{x_n \to x_0} \frac{c(x_n - x_0)}{x_n - x_0} = c.$$

(ii) $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^k$.

$$f'(x_0) = \lim_{x_n \to x_0} \frac{x_n^k - x_0^k}{x_n - x_0} = \lim_{x_n \to x_0} (x_n^{k-1} + x_n^{k-2} x_0 + \ldots + x_0^{n-1}) = kx_0^{k-1}.$$

8.2 Properties of differentiable functions

Theorem 8.2.1 (Algebra with differentiable functions). *Let $f, g : (a, b) \to \mathbb{R}$ be two functions differentiable in x_0. Then*

- $f \pm g$
- $f \cdot g$
- $\frac{f}{g}$
- $f \circ g$

is differentiable and the derivative is

- $(f \pm g)' = f' \pm g'$
- $(fg)' = f'g + fg'$(product rule *[Produktregel])*
- $\frac{f}{g} = \frac{f'g - fg'}{g^2}$ *(quotient rule [Quotientenregel])*
- $(f \circ g)' = f' \circ g \cdot g'$ *(chain rule [Kettenregel])*

Proof.

- $f \pm g$: Exercise.
- $f \cdot g$:

$$
\begin{aligned}
& \lim_{x_n \to x_0} \frac{(fg)(x_n) - (fg)(x_0)}{x_n - x_0} \\
= & \lim_{x_n \to x_0} \frac{f(x_n)g(x_n) - f(x_0)g(x_0)}{x_n - x_0} \\
= & \lim_{x_n \to x_0} \frac{f(x_n)g(x_n) - f(x_0)g(x_0) + \overbrace{f(x_n)g(x_0) - f(x_n)g(x_0)}^{=0}}{x_n - x_0} \\
= & \lim_{x_n \to x_0} \frac{f(x_n)g(x_n) - f(x_n)g(x_0) + f(x_n)g(x_0) - f(x_0)g(x_0)}{x_n - x_0} \\
= & \lim_{x_n \to x_0} \frac{f(x_n)(g(x_n) - g(x_0)) + (f(x_n) - f(x_0))g(x_0)}{x_n - x_0} \\
= & \lim_{x_n \to x_0} \frac{f(x_n)(g(x_n) - g(x_0))}{x_n - x_0} + \frac{(f(x_n) - f(x_0))g(x_0)}{x_n - x_0} \\
= & \lim_{x_n \to x_0} f(x_n)\frac{(g(x_n) - g(x_0))}{x_n - x_0} + g(x_0)\frac{(f(x_n) - f(x_0))}{x_n - x_0} \\
= & \lim_{x_n \to x_0} f(x_n) \lim_{x_n \to x_0} \frac{(g(x_n) - g(x_0))}{x_n - x_0} + \lim_{x_n \to x_0} g(x_0) \lim_{x_n \to x_0} \frac{(f(x_n) - f(x_0))}{x_n - x_0} \\
= & f(x_0)g'(x_0) + g(x_0)f'(x_0).
\end{aligned}
$$

- $\frac{f}{g}$: Exercise.
- $f \circ g$: *(we do the proof in the case where g is injective)*. Write

$$
\frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{x_n - x_0} = \frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{g(x_n) - g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0}
$$

Here we used the injectivity of g, which assures that $g(x_n) - g(x_0) \neq 0$. Now we can determine the limit:

$$
\begin{aligned}
\lim_{x_n \to x_0} \frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{x_n - x_0} &= \lim_{x_n \to x_0} \frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{g(x_n) - g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(g(x_n)) - f(g(x_0))}{g(x_n) - g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(g(x_n)) - f(g(x_0))}{g(x_n) - g(x_0)} \lim_{x_n \to x_0} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= f'(g(x_0)) \cdot g'(x_0).
\end{aligned}
$$

□

Now we shall have a look at a useful application from everday life. At first we have the following definition:

Definition 8.2.2. Let $f : (a,b) \to \mathbb{R}$ be a function and $x_0 \in (a,b)$. Then x_0 is called

- *local minimum [lokales Minimum] if there exists an* $\varepsilon > 0$ *such that for all* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ $f(x_0) \leq f(x)$
- *local maximum [lokales Maximum] if there exists an* $\varepsilon > 0$ *such that for all* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ $f(x_0) \geq f(x)$.
- *local extremum [lokale Extremstelle] if* x_0 *is either a local maximum or a local minimum.*

Now we can formulate the following important theorem:

Theorem 8.2.3 (Local extrema)**.** *Let* $f : (a,b) \to \mathbb{R}$ *be a differentiable function and* $x_0 \in (a,b)$. *If* f *has a local extremum in* x_0 *then* $f'(x_0) = 0$.

Proof. Let x_0 be a local extremum. Without loss of generality we may assume that x_0 is a local maximum; *i.e.*, $f(x_0) \geq f(x)$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Then

$$f'(x_0) = \lim_{h \to 0, h < 0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

since $f(x_0 + h) - f(x_0) \leq 0$ and $h < 0$. But on the other hand

$$f'(x_0) = \lim_{h \to 0, h > 0} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0$$

since $f(x_0 + h) - f(x_0) \leq 0$ and $h > 0$. Since f is differentiable these two limits have to coincide, which yields $f'(x_0) = 0$. □

8.3 The Mean Value Theorem

Theorem 8.3.1 (*Mean Value Theorem [Mittelwertsatz]*)**.** *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function. Then there is a real number* c, $a < c < b$, *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This theorem seems rather technical, but it is beautifully illustrated by drawing a wavy graph and showing that there is a point where the tangent has the same slope as the line through the endpoints of the graph; see Figure 8.2 for an illustration.

One should point out that c *need not be unique. Also this theorem is a typical existence theorem. It tells us that something exists, but gives us no hints how to find it. Even for simple functions it might be impossible to actually determine the value of such a* c.

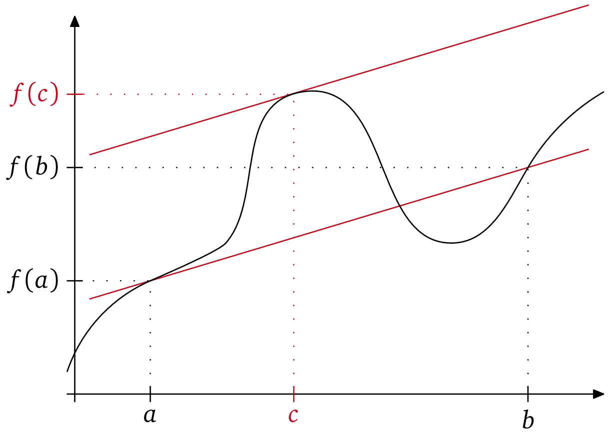

Figure 8.2: An illustration for the mean value theorem. The tangent through $(c, f(c))$ has slope $\frac{f(b)-f(a)}{b-a}$.

8.4 One application and tool: L'Hôspital's Rule

The following theorem yields another way to find limits:

Theorem 8.4.1 (*L'Hôspital's Rule [Regel von L'Hôspital]*)**.** *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions and $x_0 \in (a, b)$. Furthermore, let $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. We consider the function $\frac{f(x)}{g(x)}$.*

If $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Remark 8.4.2.

- We can only apply this rule for limits where the variable approaches a real number, *i.e.*, *not* ∞. So if we have

$$\lim_{n \to \infty} \sin\left(\frac{1}{n}\right) n,$$

then we cannot apply l'Hôspital's rule. First we have to substitute the sequence by (for example) $k := \frac{1}{n}$. As n goes to infinity, k goes to 0. This yields

$$\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{k \to 0} \frac{\sin(k)}{k}.$$

Now we can apply l'Hôspital's rule and we get

$$\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{k \to 0} \frac{\sin(k)}{k} = \lim_{k \to 0} \frac{\cos(k)}{1} = 1.$$

- Also note that it is crucial that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. Consider, for instance,

$$\lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

  but

$$\lim_{x \to 0} \frac{(\sin x)'}{(\cos x)'} = \lim_{x \to 0} \frac{\cos x}{-\sin x} = -\infty.$$

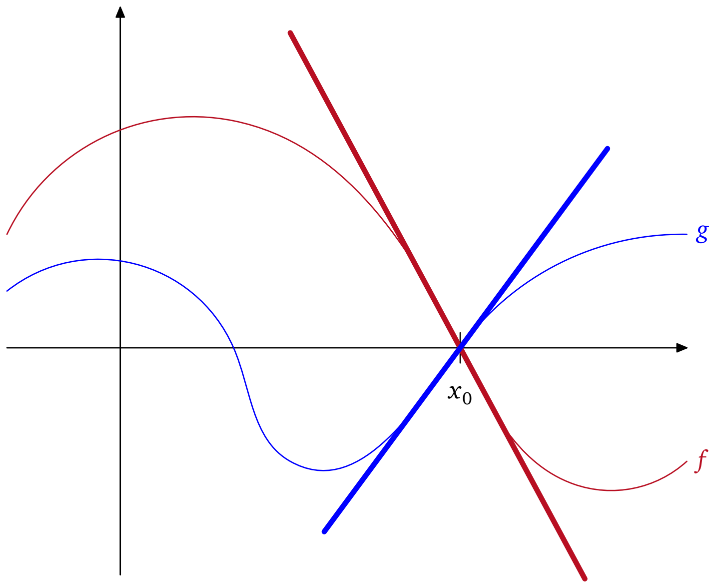

Figure 8.3: Two functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x_0) = g(x_0) = 0$. By l'Hôspital's rule, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.