Lecture 7 — Continuous Functions

We talked about functions in Chapter 4. Recall that a function $f : X \to Y$ assigns to every element x of the set X an element $y = f(x)$ of the set Y.

7.1 Everyday examples

Example 7.1.1.

(i) If we drive a car with constant velocity v then we know from physics that the distance s we travelled at time t equals

$$s = s_0 + v \cdot t,$$

where s_0 is the initial distance at time $t = 0$. We see immediately that s depends on the time t, *i.e.*, s is a function in t.

$$\begin{aligned} s : [0,\infty) &\longrightarrow \mathbb{R} \\ t &\longmapsto s_0 + v \cdot t \end{aligned}$$

The graph of this function is illustrated in Figure 7.1 (a).

We see that small changes of the variable t induce small changes in the distance s.

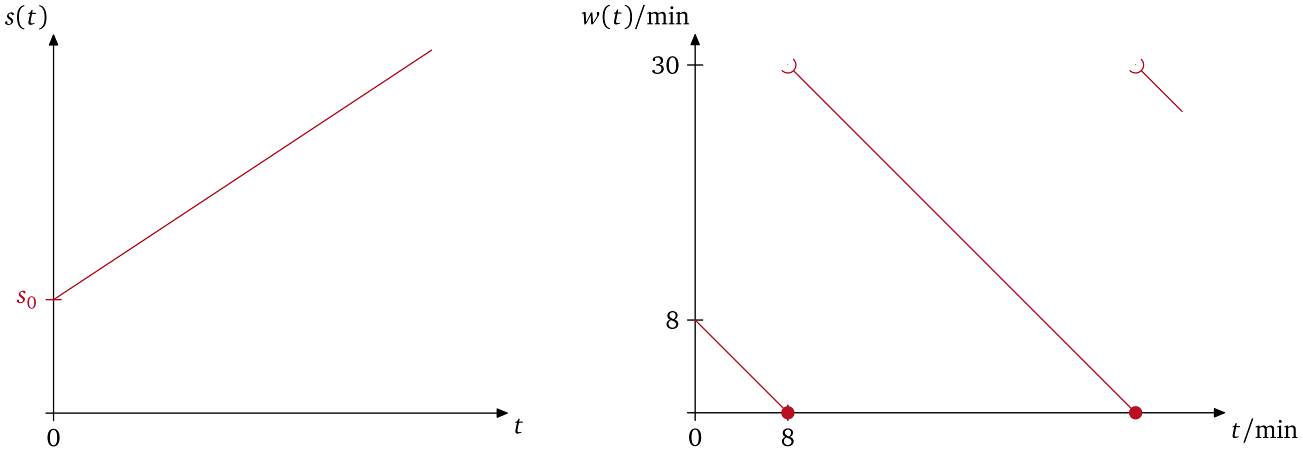
(ii) When we use the train from Darmstadt to Frankfurt and we want to catch another train in Frankfurt, then we are interested in the delay of the first train. We look at the function

$$w : [0,\infty) \longrightarrow [0,\infty)$$

which assignes to each delay t the time $w(t)$ we have to wait in Frankfurt. If we assume the first train to arrive at $xx : 48$ and the next train to depart at $xx : 56$, then a delay of 3 minutes means that we have to wait 5 minutes. And a delay of 8 minutes means that we have to wait 0 minutes. But if we had a delay of $8 + \varepsilon$ minutes, then we will not catch the train and we will have to wait for the next one (assume it departs in 30 minutes).

So a delay of 8 minutes means no waiting time. But if the delay is only a little bit more than that we will have to wait for almost 30 minutes. This means a small change in the variable t can result in a big change of the variable w.

The graph of this function is illustrated in Figure 7.1 (b).



(a) The function $s : [0,\infty) \to \mathbb{R}$ in (i) is *continuous*.

(b) The function $w : [0,\infty) \to [0,\infty)$ in (ii) is *not continuous*.

Figure 7.1: The graphs of the functions in Example 7.1.1.

7.2 Definitions of continuity

Definition 7.2.1 (ε-δ-condition)**.** A function $f : X \to Y$ is called *continuous [stetig] in a point $x_0 \in X$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\left|f(x) - f(x_0)\right| < \varepsilon \qquad \text{for all } x \in X \text{ with } \left|x - x_0\right| < \delta. \tag{7.1}$$

Equivalently, using quantifiers, this reads: $f : X \to Y$ is continuous in $x_0 \in X$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X) \left|x - x_0\right| < \delta \Rightarrow \left|f(x) - f(x_0)\right| < \varepsilon.$$

Now let us have another look at our examples:

(i) *Fix some value t_0. We have*

$$\left|s(t_0) - s(t)\right| = \left|s_0 + v \cdot t_0 - (s_0 + v \cdot t)\right| = \left|v \cdot t_0 - v \cdot t\right| = \left|v(t_0 - t)\right| = |v| \cdot \left|t_0 - t\right|.$$

Now choose $\varepsilon > 0$ arbitrarily and assume that $\left|s(t_0) - s(t)\right| < \varepsilon$. We have to find $\delta > 0$ such that $\left|s(t_0) - s(t)\right| < \varepsilon$ holds for all t with $\left|t_0 - t\right| < \delta$. From the inequality

$$\left|s(t_0) - s(t)\right| = |v| \cdot \left|t_0 - t\right| < \varepsilon$$

we see that we can pick $\delta := \frac{\varepsilon}{|v|}$ and we can easily verify that this will work. This proves that s is continuous.

Note also that δ may depend on ε.

(ii) *Consider $t_0 = 8$. We will show that w is not continuous in t_0. I.e., we have to show that there is some $\varepsilon > 0$ such that for each $\delta > 0$ there is some t with $\left|t_0 - t\right| < \delta$ but $\left|w(t_0) - w(t)\right| > \varepsilon$.*

Choose $\varepsilon = 1$ and let $\delta > 0$. Set $t := t_0 + \min\left\{1, \frac{\delta}{2}\right\}$. Then $\left|t_0 - t\right| = \min\left\{1, \frac{\delta}{2}\right\} < \delta$ but $\left|w(t_0) - w(t)\right| = |w(t)| = 30 - \min\left\{1, \frac{\delta}{2}\right\} > 1$.

Hence, w is not continuous at $t_0 = 8$.

If we have a function $\mathbb{R} \to \mathbb{R}$, then there is another characterisation of continuity:

Theorem 7.2.2 (Limit test)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on a neighbourhood U of $x_0 \in \mathbb{R}$ but not necessarily defined in x_0. Then there are equivalent:*

(a) *f is continuous in x_0,*

(b) *for every convergent sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with limit x_0 we have that*

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x_0).$$

Proof. "$\Rightarrow$" Assume that the ε-δ-condition is satisfied in x_0. We need to show that for any sequence $x_n \to x_0$ in the domain, the image sequence satisfies $f(x_n) \to f(x_0)$. Let $\varepsilon > 0$ be arbitrary, and pick a $\delta > 0$ as in Equation (7.1) on page 2. Since $x_n \to x_0$, we can choose $N \in \mathbb{N}$ such that

$$\left|x_n - x_0\right| < \delta \qquad \text{for all } n \geq N.$$

But then Equation (7.1) implies

$$\left|f(x_n) - f(x_0)\right| < \varepsilon \qquad \text{for all } \left|x_n - x\right| < \delta,$$

which is equivalent to

$$\left|f(x_n) - f(x_0)\right| < \varepsilon \qquad \text{for all } n \geq N,$$

which implies that $f(x_n)$ converges to $f(x_0)$.

"$\Leftarrow$" Assume that the limit condition holds. We prove the continuity of f by contradiction.

Suppose there exists $\varepsilon > 0$ for which we cannot find $\delta > 0$ such that Equation (7.1) holds. In particular, (7.1) will not be satisfied for $\delta = \frac{1}{n}$ for any $n \in \mathbb{N}$. Thus, there exists x_n with $\left|x_n - x_0\right| < \frac{1}{n}$ such that

$$\left|f(x_n) - f(x_0)\right| > \varepsilon.$$

Hence $f(x_n)$ does not converge to $f(x_0)$. □

Example 7.2.3.

(i) Consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$. Choose $x_0 = 0$. Then the function has the limit 0 in x_0. For a proof, we need to choose any sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = 0$. Now we have to show that the sequence $f_n = x_n^2$ for $n \in \mathbb{N}$ converges to 0. This, however, is not difficult using our theorem about algebra with sequences:

$$\lim_{n\to\infty} x^2 = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = 0 \cdot 0 = 0.$$

(ii) Now let us look at a more complicated example which does not have any limit in 0. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto \sin\left(\frac{1}{x}\right)$.

Consider the sequence $x_n = \frac{1}{\pi \cdot n}$ for $n \in \mathbb{N}$. Then

$$f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin(\pi \cdot n) = 0.$$

Now consider the sequence $y_n = \frac{2}{\pi \cdot (4n+1)}$. This again gives a sequence of function values:

$$f(y_n) = \sin\left(\frac{1}{y_n}\right) = \sin\left(\frac{\pi \cdot (4n+1)}{2}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$$

So this time we get 1 as the limit of our sequence.

For two different sequences we have obtained two different limits. Therefore, by the limit test (Theorem 7.2.2), this function is not continuous in 0.

7.3 Properties of continuous functions

Theorem 7.3.1 (Algebra with continuous functions)**.** *Let $f, g : U \to \mathbb{R}$ be two continuous functions. Then*

- $f \pm g$,
- $f \cdot g$,
- $\frac{f}{g}$, *and*
- $f \circ g$

are continuous (where they are defined).

Proof. This will be an exercise for you :-). □

Example 7.3.2. The following are examples of continuous functions:

(i) All polynomials are continuous. This follows easily from the theorem about algebra with continuous functions and from the fact that the constant functions and the identity function on $\mathbb{R}$ are continuous.

(ii) We define the following functions:

$$\begin{aligned}
\exp : \mathbb{R} &\longrightarrow \mathbb{R} \\
x &\longmapsto \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin : \mathbb{R} &\longrightarrow \mathbb{R} \\
x &\longmapsto \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos : \mathbb{R} &\longrightarrow \mathbb{R} \\
x &\longmapsto \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

These functions are continuous.

(iii) Rational functions are continuous on the subset of $\mathbb{R}$ where the denominator is different from 0.

The following is an important theorem about continuous functions:

Theorem 7.3.3 (*Intermediate Value Theorem [Zwischenwertsatz]*). *Let $f : [a,b] \to \mathbb{R}$ be continuous and let c be strictly between $f(a)$ and $f(b)$. Then there is an x strictly between a and b such that $f(x) = c$.*

We will not give a proof because it is very technical. See Figure 7.2 for an illustration.

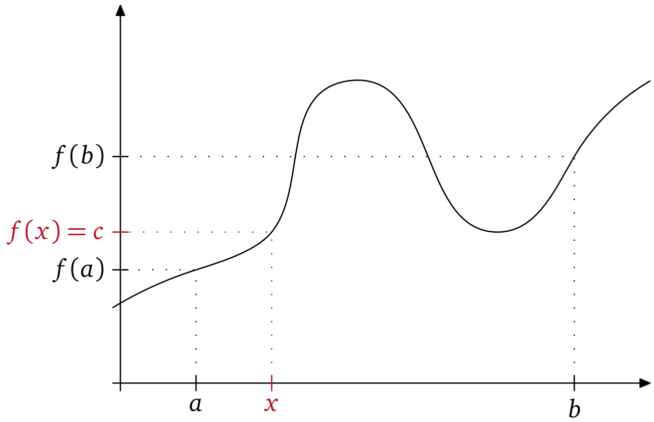


Figure 7.2: f is a continuous function $[a,b] \to \mathbb{R}$. Moreover, $f(a) < c < f(b)$ is chosen. By the intermediate value theorem there is $x \in [a,b]$ such that $f(x) = c$.