# Lecture 6 — Series

## 6.1 Partial sums and convergence

**Definition 6.1.1.**

(i) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Then a *series [Reihe] is the sequence* $(s_n)_{n\in\mathbb{N}}$ *of partial sums [Partialsummen]*

$$s_n := a_1 + \ldots + a_n.$$

*Usually we write* $\sum_{n=1}^{\infty} a_n$ *for the sequence* $(s_n)_{n\in\mathbb{N}}$ *and call* $a_n$ *its terms [Summanden].*

(ii) *In case the series* $(s_n)_{n\in\mathbb{N}}$ *converges to* $s \in \mathbb{R}$ *we write*

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s.$$

**Remark 6.1.2.** In the convergent case, the notation $\sum_{n=1}^{\infty} a_n$ has two different meanings:

- The sequence of partial sums $(a_1 + \ldots + a_n)_{n\in\mathbb{N}}$, and
- a number $s \in \mathbb{R}$, namely the limit of the partial sums; it is also called the *value [Wert] of the series.*

**Example 6.1.3.**

(i) *Decimal expansion [Dezimaldarstellung]: The decimal expansion of a number* $x \in \mathbb{R}$ *can be defined as a series. The partial sums are* $s_n = d_0 + \frac{d_1}{10} + \ldots + \frac{d_n}{10^n}$*, that is, finite expansions up to the* $n$*-th digit. The limit is* $x = \lim s_n$*. For instance* $\pi = 3.14\ldots = 3 + \frac{1}{10} + \frac{4}{100} + \ldots$*.*

(ii) *We claim* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$*, i.e., we claim for the sequence* $s_n$ *of partial sums that*

$$s_n := \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} \quad \to \quad 1 \quad \textit{as} \quad n \to \infty.$$

*Proof: Writing*

$$\frac{1}{n(n+1)} = \frac{-(n^2-1)+n^2}{n(n+1)} = -\frac{n-1}{n} + \frac{n}{n+1}, \qquad \textit{for } n \in \mathbb{N},$$

*we see we can apply a telescope sum trick:*

$$s_n = \Big(-0 + \frac{1}{2}\Big) + \Big(-\frac{1}{2} + \frac{2}{3}\Big) + \Big(-\frac{2}{3} + \frac{3}{4}\Big) + \ldots + \Big(-\frac{n-1}{n} + \frac{n}{n+1}\Big)$$
$$= -0 + \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \quad \to \quad 1 \qquad \textit{as } n \to \infty.$$

**Example 6.1.4.** If we are careless, we can easily run into contradictions:

$$0 = (1-1) + (1-1) + \ldots = 1 + (-1+1) + (-1+1) + \ldots = 1.$$

*In naive language, infinite sums are not associative.*

*The following theorem give some necessary condition on a sequence in order for the corresponding series to converge:*

**Theorem 6.1.5.** *If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. (We also write $a_n \to 0$ as $n \to \infty$.)*

*Proof.* We have $a_n = s_n - s_{n-1}$ for $n \geq 2$ and thus, using $s_n = \sum_{k=1}^{n} a_k \to s$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$ □

## 6.2 Important examples

*In Theorem 6.1.5 we have seen that the summands of a convergent series form a null sequence.*

*The converse, however, does not hold as the following example shows:*

**Example 6.2.1.** The *harmonic series [harmonische Reihe]*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

*has terms $a_n = \frac{1}{n}$ forming a null sequence.*

*Nevertheless, the sequence of partial sums is unbounded. Indeed, for $n \geq 1$ consider the subsequence [Teilfolge]*

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \\ &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq 1/2} + \underbrace{\left(\frac{1}{5} + \ldots + \frac{1}{8}\right)}_{\geq 1/2} + \ldots + \underbrace{\left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right)}_{\geq 1/2} \\ &\geq 1 + \frac{n}{2} \to \infty. \end{aligned}$$

*Thus, the harmonic series does not converge.*

*Moreover $(s_n)_{n\in\mathbb{N}}$ is increasing, and hence our argument shows that $\sum \frac{1}{n}$ diverges to infinity; as for sequences we denote this symbolically by $\sum \frac{1}{n} = \infty$.*

*A very important series will turn out to be the following:*

**Theorem 6.2.2.** *Let $x \in \mathbb{R}$. The* geometric series *[geometrische Reihe]*

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$$

*converges for all $|x| < 1$ to*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

*while for $|x| \geq 1$ the series diverges.*

*Proof.* The geometric sum gives

$$s_n = \sum_{j=0}^{n} x^j = 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \qquad \text{for } x \neq 1. \tag{6.1}$$

When $|x| < 1$ we see that $x^n \to 0$ as $n \to \infty$; hence $\lim s_n = \frac{1}{1-x}$.

For $|x| \geq 1$ also $|x^n| = |x|^n \geq 1$, and so $(x^n)$ is not a null sequence and hence $\sum x^n$ diverges by Theorem 6.1.5. □

**Example 6.2.3.**

- $\left|\frac{1}{2}\right| < 1$ and hence
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2.$$
- $\left|\frac{1}{3}\right| < 1$ and hence
$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}.$$
- $\left|-\frac{1}{2}\right| < 1$ and hence
$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \pm \ldots = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

**Example 6.2.4.** A periodic decimal expansion is, up to an additive constant, a geometric series; it always defines a rational number. For example,

$$\begin{aligned} 2.\overline{34} :=& 2.343434\ldots = 2 + \frac{34}{10^2} + \frac{34}{10^4} + \frac{34}{10^6} + \cdots = 2 + \frac{34}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \ldots\right) \\ =& 2 + \frac{34}{100} \cdot \frac{1}{1 - \frac{1}{100}} = 2 + \frac{34}{100} \cdot \frac{100}{99} = 2 + \frac{34}{99} = \frac{232}{99}. \end{aligned}$$

## 6.3 Series of real numbers

*In this section we will give some criteria to test for convergence of series.*

**Theorem 6.3.1.** *A series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ converges if and only if its partial sums are bounded.*

*Proof.* The assumption $a_n \geq 0$ means that the sequence of partial sums $(s_n)$ is increasing. Therefor $s_{n+1} \geq s_n$ and so $s_n \leq s$ and hence $\sum_{n=1}^{\infty} a_n$ converges by Theorem 5.3.5. □

**Example 6.3.2.** Consider a decimal expansion $0.d_1d_2d_3\ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ with $d_n \in \{0,1,\ldots,9\}$. The partial sums

$$s_n = \frac{d_1}{10} + \frac{d_2}{100} + \ldots + \frac{d_n}{10^n}$$

are increasing in $n$ and are bounded by

$$s_n \leq \frac{9}{10} + \frac{9}{100} + \ldots + \frac{9}{10^n} \overset{\text{geom.series}}{=} \frac{9}{10} \cdot \frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}} < \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9} = 1$$

(our estimate says that $0.99\ldots9$, with $n$ digits, is indeed less than 1).

Thus, by Theorem 6.3.1 every decimal expansion converges.

*This boundedness criterion can be used for a comparison test for convergence:*

**Theorem 6.3.3** (Majorisation of real series)**.** *Suppose $(x_n)_{n\in\mathbb{N}}$ is a real sequence for which there exists a convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers $a_n \geq 0$ with*

$$0 \leq x_n \leq a_n \qquad \text{for all } n \in \mathbb{N}.$$

*Then $\sum_{n=1}^{\infty} x_n$ also converges and $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} a_n$.*

*We say that $a_n$ majorises [majorisiert] $x_n$.*

*Proof.* Denote $C := \sum_{k=1}^{\infty} a_k$. We consider partial sums. By assumption, $\sum_{k=1}^{n} a_k \leq C$ and so

$$0 \leq \sum_{k=1}^{n} x_k \leq \sum_{k=1}^{n} a_k \leq C.$$

Thus, $\sum_{k=1}^{\infty} x_k$ converges by Theorem 6.3.1. □

**Exercise 6.3.4.** Suppose that for a real series $\sum_{n=1}^{\infty} a_n$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $a_n \geq x_n \geq 0$ such that $\sum_{n=1}^{\infty} x_n$ is divergent. Prove that $\sum_{n=1}^{\infty} a_n$ diverges as well.

## 6.4 Trigonometric and other functions

*Many important functions can be defined as series:*

- *The exponential function [Exponentialfunktion]:*

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- *The sine function [Sinusfunktion]:*

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- *The cosine function [Cosinusfunktion]*

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$