Lecture 5 — Sequences

5.1 Examples and definition

The mathematical concept of a *sequence [Folge] is easy to understand. First we look at a few examples.*

Example 5.1.1.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots \quad \text{the sequence of natural numbers}$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \quad \text{a sequence of rational numbers}$$

$$-1, 1, -1, 1, -1, 1, -1, \ldots \quad \text{a sequence of 1s and } -1\text{s}$$

$$\pi, \frac{2}{3}, 15, \log 2, \sqrt{15}, \ldots \quad \text{a sequence of random real number}$$

The characteristic feature of a sequence of numbers is the fact that there is a first term of the sequence, a second term, and so on. In other words, the numbers in a sequence come in a particular order. This gives rise to the following formal definition:

Definition 5.1.2. A sequence of real numbers is a map from the natural numbers $\mathbb{N}$ into the set $\mathbb{R}$ of real numbers. This means that for each natural number n there is an element of the sequence, which we denote by a_n. In this notation, the elements of the sequence can be listed as

$$a_1, a_2, a_3, \ldots$$

More concisely, we write $(a_n)_{n\in\mathbb{N}}$ for the sequence.

Example 5.1.3.

(i) Let c be a fixed constant real number. Then the sequence $a_n = c$ for $n \in \mathbb{N}$ is called *constant sequence [konstante Folge].*

$$c, \quad c, \quad c, \quad c, \quad c, \quad c, \quad c, \quad \ldots$$

(ii) $a_n = \frac{1}{n}$ *for* $n \in \mathbb{N}$. *The term* a_{17} *is* $\frac{1}{17}$. *A sequence like this is defined explicitly. It is given by a formula which can be used directly to compute an arbitrary term of the sequence.*

(iii) *Here is another example of an explicitly given sequence:* $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$.

(iv) *Define* $a_1 = 1$ *and* $a_{n+1} = a_n + (2n+1)$. *This is a recursively [rekursiv] defined sequence. To compute* a_{n+1} *we need to know* a_n, *for which we need to know* a_{n-1} *and so on. Sometimes it is not difficult to find an explicit description for a recursively defined sequence. In this case we have* $a_n = n^2$.

(v) *A famous (and more difficult) example for a recursively defined sequence is the Fibonacci sequence:* $f_1 = 1$, $f_2 = 1$ *and* $f_{n+1} = f_n + f_{n-1}$ *for* $n > 2$, $n \in \mathbb{N}$. *The first few terms of the sequence are*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

There is the following closed form:

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

5.2 Limits and convergence

We will now have a closer look at the terms of the sequences (ii) and (iii) from Example 5.1.3 above:

$$1, \quad \frac{1}{2} = 0.5, \quad \frac{1}{3} = 0.3333\ldots, \quad \frac{1}{4} = 0.25, \frac{1}{5} = 0.2, \quad \ldots, \quad \frac{1}{200} = 0.005, \quad \ldots$$

$$\frac{1}{2} = 0.5, \quad \frac{2}{3} = 0.6666\ldots, \quad \frac{3}{4} = 0.75, \quad \frac{4}{5} = 0.8, \quad \ldots, \quad \frac{199}{200} = 0.995, \quad \ldots$$

While the terms of the first sequence get closer and closer to 0, *the terms of the second sequence get closer and closer to* 1. *Although no term of either sequence ever reaches* 0 *or* 1, *respectively, we would like to be able to express the fact that both sequences approach a certain number and get arbitrarily close.*

Definition 5.2.1. A sequence $(a_n)_n \in \mathbb{N}$ has a *limit [Grenzwert]* $a \in \mathbb{R}$ *if for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$\left| a_n - a \right| < \varepsilon \text{ for } n \geq N.$$

If the sequence $(a_n)_{n\in\mathbb{N}}$ *has a limit* a, *then* $(a_n)_{n\in\mathbb{N}}$ *is called* convergent *[konvergent] and we write*

$$\lim_{n\to\infty} a_n = a.$$

Read: The limit of a_n *as* n *goes to* ∞ *is* a.

If a sequence is not convergent it is called divergent *[divergent].*

It is worthwhile to think about this definition for a while and understand what the different parts of the definition mean. One way to interpret it is to say that a *is a limit of a sequence* $(a_n)_{n\in\mathbb{N}}$ *if the distance of* a *to all except a finite number of terms of the sequence is smaller than* ε. *The finite number of terms which may be further away from* a *than* ε *are*

$$a_1, a_2, a_3, \ldots, a_{N-1}.$$

Note that N *depends on* ε, *although we do not explicitly state this in the definition. This is because we have to choose* N *appropriately, depending on the given* ε.

Example 5.2.2. Let us consider the sequence $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. We would like to show that the sequence has limit 0. We will follow the definition of a limit and need to show that for each $\varepsilon > 0$ there is an N such that

$$\left|\frac{1}{n}\right| < \varepsilon \text{ for all } n \geq N.$$

We take ε as given. The condition $\frac{1}{n} < \varepsilon$ is equivalent to the condition $n > \frac{1}{\varepsilon}$. So let us try to choose N to be the next natural number larger than $\frac{1}{\varepsilon}$. Then we have that $\frac{1}{N} < \varepsilon$. With this we get the following chain of inequalities for $n \geq N$:

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

In particular, we see that $a_n = \frac{1}{n} < \varepsilon$ for all $n \geq N$. Hence we have shown that 0 is the limit of the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$.

Example 5.2.3. The sequence $1, -1, 1, -1, \ldots$ is divergent. It is interesting to prove this using the definition of limit. It requires working (implicitly or explicitly) with the negation of the defining property including the various quantifiers.

Theorem 5.2.4 (Algebra with sequences). *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences. Then:*

(i) *$(a_n \pm b_n)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

(ii) *$(a_n \cdot b_n)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

(iii) *If $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$ then $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Exercise 5.2.5. What happens if one of the sequences (for example sequence $(a_n)_{n\in\mathbb{N}}$) is divergent? What can we say about

$$(a_n + b_n)_{n\in\mathbb{N}},$$
$$(a_n \cdot b_n)_{n\in\mathbb{N}} \quad \text{and}$$
$$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}?$$

5.3 One test for convergence

Definition 5.3.1. We will say that a sequence is *increasing [steigend] if* $a_n \geq a_m$ *whenever* $n > m$. *It is decreasing [fallend] if* $a_n \leq a_m$ *whenever* $n > m$. *A sequence is monotone [monoton] if it is either increasing or decreasing.*

Example 5.3.2.

- The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is decreasing since $\frac{1}{n} < \frac{1}{m}$ if $n > m$.
- The sequence $\left(\frac{n-1}{n}\right)_{n\in\mathbb{N}}$ is increasing.
- The sequence $((-1)^n)_{n\in\mathbb{N}}$ is neither increasing nor decreasing.

Definition 5.3.3. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ is *bounded [beschränkt] if there are some* $b_u, b_l \in \mathbb{R}$ *such that* $b_l \leq a_n \leq b_u$ *for all* n. *In this case we call* b_u *and* b_l *the upper [obere Schranke] and lower bound [untere Schranke], respectively.*

Otherwise we say that $(a_n)_{n\in\mathbb{N}}$ *is unbounded [unbeschränkt].*

Example 5.3.4.

- The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is bounded since $0 \leq \frac{1}{n} \leq 1$ for each $n \in \mathbb{N}$.
- The sequence $(2^n)_{n\in\mathbb{N}}$ is unbounded since for each $b \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $2^n > b$. *I.e.*, there is no upper bound.

Theorem 5.3.5. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a monotone sequence. If* a_n *is bounded then* a_n *converges.*