Lecture 4 — Functions

Definition 4.0.1. Let X and Y be sets. A *function [Funktion] f from the set X to the set Y, denoted $f: X \to Y$, is a rule that assigns to each element of X exactly one element of Y.*

The element of Y assigned to a particular element $x \in X$ is denoted by $f(x)$ and is called the image of x under f [Bild von x unter f]. Vice versa, x is called a preimage [Urbild] of $y = f(x)$. Note that an element $y \in Y$ can have more than one preimage under f or may not a have a preimage at all.

The set X is called the domain [Definitionsbereich] of f and Y is called the range [Wertebereich] of f. The set $\{f(x) \mid x \in X\}$ of all images is called the image [Bild] of f.

It is important to understand that the domain and the range are an essential part of the definition of a function. For example, consider the functions

$$\begin{array}{lll} f: & \mathbb{R} \to \mathbb{R} & x \mapsto x^2 \\ g: & \mathbb{R} \to \mathbb{R}_{\geq 0} & x \mapsto x^2 \end{array}$$

Strictly speaking, these are two different functions. One obvious difference is that all elements in the range of g do have a preimage, while there are elements in the range of f which do not have a preimage (-1 for example). So the statement "All elements in the range have a preimage." is true for g and false for f.

Example 4.0.2.

(i) Let $c \in Y$ be constant. Then the function

$$\begin{array}{rrcl} f: & X & \to & Y \\ & x & \mapsto & c \end{array}$$

is called a *constant function [konstante Funktion]. It maps each element of X to the same value c.*

(ii) *The function*

$$\begin{array}{rrcl} \mathrm{id}_X: & X & \to & X \\ & x & \mapsto & x \end{array}$$

is called the identity function [Identität] of X. It maps each element of X to itself.

4.1 Properties of functions

Definition 4.1.1. Let $f: X \to Y$ be a function.

- The function f is called *injective [injektiv] iff for all $x_1, x_2 \in X$*

$$x_1 = x_2 \iff f(x_1) = f(x_2).$$

- *The function f is called surjective [surjektiv] iff for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.*
- *If f is injective and surjective then it is bijective [bijektiv], i.e., f is bijective iff for each $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$.*

Example 4.1.2.

(i) The function id_X is bijective.

(ii) The constant function $f : x \mapsto c$ for a fixed c is injective if and only if X has exactly one element. It is surjective if and only if Y has exactly one element.

(iii) The function

$$f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x(x-1)(x+1) \end{array}$$

is not injective because $f(-1) = f(0) = f(1) = 0$. The function is surjective because the equation $f(x) = c$ is equivalent to the equation $x^3 - x - c = 0$, which is a polynomial of degree three, which has a zero in $\mathbb{R}$.

4.2 Algebra with functions

Definition 4.2.1. We consider functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then we can construct new functions

(i) $f \pm g : x \mapsto f(x) \pm g(x)$ for $x \in X \cap Y$,

(ii) $f \cdot g : x \mapsto f(x) \cdot g(x)$ for $x \in X \cap Y$,

(iii) $\frac{f}{g} : x \mapsto \frac{f(x)}{g(x)}$ for $x \in X \cap Y$ and $g(x) \neq 0$,

(iv) $g \circ f : x \mapsto g(f(x))$ if $f(X)$ is contained in Y.

This is called the *composition [Hintereinanderausführung/Verkettung] of functions. The function f is the inner function [innere Funktion] and the function g is the outer function [äußere Funktion].*

Example 4.2.2. Consider the function $f : \mathbb{R} \to \mathbb{R} : f(x) = \sqrt{x^2 + 1}$ and decompose it as follows: Let $\mathbf{1}_{\mathbb{R}} : x \mapsto 1$ and $\sqrt{\cdot} : x \mapsto \sqrt{x}$. Then

$$f = \sqrt{\cdot} \circ (\mathrm{id}_{\mathbb{R}} \cdot \mathrm{id}_{\mathbb{R}} + \mathbf{1}_{\mathbb{R}})$$

Theorem 4.2.3. *Let $f : X \to Y$ be a bijective function. Then there is a unique function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.*

The function g is called the *inverse function [Umkehrfunktion] of f. We write $g = f^{-1}$. If $f(x) = y$, then $f^{-1}(y) = x$.*

4.3 Types of functions on $\mathbb{R}$

The following is a list of certain frequently appearing types of functions on $\mathbb{R}$.

constant functions *Let $c \in \mathbb{R}$. Then a function $f(x) = c$ is a constant function.*

power functions *The function $f(x) = x^n$ for a natural number n is called a power function [Potenzfunktion].*

polynomials *A function of the form $f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$ is called a polynomial function [Polynom]. Polynomial functions are built from the identity function $\mathrm{id}_{\mathbb{R}}$ and the constant functions using $+, -, \cdot$.*

rational functions *A function of the form $f(x) = p(x)/q(x)$ with polynomials p and q is called a rational function [rationale Funktion]. Note that its maximal domain is $\mathbb{R} \setminus \{x \in \mathbb{R} \mid q(x) = 0\}$.*

algebraic functions *Algebraic functions [algebraische Funtionen] are constructed from polynomials (or, equivalently from the identity function and the constant functions) by using $+$, $-$, $\cdot$, $/$ and taking roots.*