# Lecture 2 — Propositional Logic

Examples of propositions: 5 is not a number. Darmstadt is in Germany. Mathematics is a science. 7 divides 12.

*A proposition [Aussage] is a grammatically correct statement which it can be decided of whether it is true or false.*

*More interesting than deciding whether one proposition is true or false is to decide whether a proposition is true under certain circumstances. This process is fundamental in mathematics.*

*We now have a look at how to combine given propositions to new propositions and under which circumstances the new proposition is true.*

## 2.1 Logical operators

*Whenever we construct a new proposition from other propositions we can use a truth table [Wahrheitstabelle] to describe the newly constructed proposition. We simply write down the value of the new one for all combinations of values of the old ones. This section gives several examples for truth tables.*

***Negation:*** *The negation [Verneinung] of a proposition $A$ is false when $A$ is true and vice versa (written $\neg A$):*

| $A$ | $\neg A$ |
|---|---|
| $t$ | $f$ |
| $f$ | $t$ |

***And (Conjunction):*** *Two propositions $A$ and $B$ can be combined by "and" to give a new proposition $A \wedge B$ (the conjunction [Konjunktion] of $A$ and $B$) which is true precisely when both $A$ and $B$ are true:*

| $A$ | $B$ | $A \wedge B$ |
|---|---|---|
| $t$ | $t$ | $t$ |
| $t$ | $f$ | $f$ |
| $f$ | $t$ | $f$ |
| $f$ | $f$ | $f$ |

***Or (Disjunction):*** *Two propositions $A$ and $B$ can be combined by "or" to give a new proposition $A \vee B$ (the disjunction [Disjunktion] of $A$ and $B$) which is true precisely when at least one of $A$ and $B$ is true:*

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| $t$ | $t$ | $t$ |
| $t$ | $f$ | $t$ |
| $f$ | $t$ | $t$ |
| $f$ | $f$ | $f$ |

***Implication:*** *If a proposition $B$ is true whenever another proposition $A$ is true, then $A$ implies [impliziert] $B$. We write $A \Rightarrow B$ and call this statement an implication [Implikation]:*

| $A$ | $B$ | $A \Rightarrow B$ |
|---|---|---|
| $t$ | $t$ | $t$ |
| $t$ | $f$ | $f$ |
| $f$ | $t$ | $t$ |
| $f$ | $f$ | $t$ |

***Equivalence:*** *A proposition $A$ is equivalent [äquivalent] to a proposition $B$ (written $A \Longleftrightarrow B$) if $A$ is true precisely when $B$ is true and $A$ is false precisely when $B$ is false (also written $A$ iff $B$, which means $A$ is true if and only if $B$ is true).*

| $A$ | $B$ | $A \Longleftrightarrow B$ |
|---|---|---|
| $t$ | $t$ | $t$ |
| $t$ | $f$ | $f$ |
| $f$ | $t$ | $f$ |
| $f$ | $f$ | $t$ |

*We give another characterisation for equivalence. And we take this as an example for a typical proof of such logical propositional statements:*

**Theorem 2.1.1.** *Let $A$ and $B$ be two propositions. Then the following are equivalent:*

(i) $((A \Rightarrow B) \wedge (B \Rightarrow A))$

(ii) $(A \Longleftrightarrow B)$

*Proof.*

| $A$ | $B$ | $A \Rightarrow B$ | $B \Rightarrow A$ | $(A \Rightarrow B) \wedge (B \Rightarrow A)$ |
|---|---|---|---|---|
| $t$ | $t$ | $t$ | $t$ | $t$ |
| $t$ | $f$ | $f$ | $t$ | $f$ |
| $f$ | $t$ | $t$ | $f$ | $f$ |
| $f$ | $f$ | $t$ | $t$ | $t$ |

□

## Implications that are not equivalences

*Here are some examples for implications which are only true "in one direction", i.e., they are no equivalences:*

- *For all $x \in \mathbb{R}$: $x > 0 \Rightarrow x^2 > 0$.*
- *If $x$ and $y$ are negative real numbers, then $x \cdot y > 0$.*

*Now let us have a look at the converses [Umkehrung] of the above propositions:*

- *For all $x \in \mathbb{R}$: $x^2 > 0$ then $x > 0$.*
- *If $x \cdot y > 0$ then $x$ and $y$ are negative real numbers.*

*In both cases we easily find a counterexample to refute these propositions.*

## 2.2 Quantifiers

*Another important feature of propositional logic are the quantifiers [Quantoren]:*

***All quantifier** [All-Quantor]: If for each element $e$ of a set $S$ a proposition $A(e)$ is given then*

$$\forall e \in S : A(e)$$

*is a proposition which is true iff $A(e)$ is true for each $e \in S$. (Read: For all $e$ in $S$, $A(e)$ is true.)*

***Existence quantifier** [Existenz-Quantor]: If for each element $e$ of a set $S$ a proposition $A(e)$ is given then*

$$\exists e \in S : A(e)$$

*is a proposition which is true iff $A(e)$ is true for at least one $e \in S$. (Read: It exists an element $e$, such that $A(e)$ is true.)*

## 2.3 Negation of propositions

*Consider the following examples:*

- *All sheep are black.*
- *It exists a male student at the TU Darmstadt.*
- *An animal is a lion or a duck.*
- *A real number is positive and negative.*

*Now consider the negations of the above:*

- *If all sheep are black, then there is no sheep with another colour. So the negation is: There exists a sheep which is not black.*
- *This proposition is true if at least one student at the TU Darmstadt is male. So the negation is: All students at the TU Darmstadt are not male.*
- *To be true, each animal has to be a duck or a lion. So the negation is: There is an animal which is neither a duck nor a lion.*
- *The proposition is true if all real numbers are both positive and negative. So the negation is: There is a real number which is not negative or not positive.*

*We summarise these insights using quantifier symbols:*

**Proposition 2.3.1.**

- $\neg(\forall e \in S : A(e)) \quad \Longleftrightarrow \quad \exists e \in S : \neg A(e)$
- $\neg(\exists e \in S : A(e)) \quad \Longleftrightarrow \quad \forall e \in S : \neg A(e)$
- $\neg(A \vee B) \quad \Longleftrightarrow \quad \neg A \wedge \neg B$
- $\neg(A \wedge B) \quad \Longleftrightarrow \quad \neg A \vee \neg B$

*As a rule of thumb you can keep in mind that a negation moving to and fro inside a proposition flips every quantifier it passes.*

## 2.4 The order of quantifiers

*It is very important to understand that we cannot change the order of the quantifiers in a proposition. E.g. let* $A(x,y)$ *be a proposition defined for each* $x$ *and* $y$*. Then the proposition* $(\forall x)(\exists y) : A(x,y)$ *is not equivalent to the proposition* $(\exists y)(\forall x) : A(x,y)$*.*

*This is immediately evident if you consider the following everyday example:*

**Example 2.4.1.** Consider the following proposition:

$$(\forall p \in \text{people})(\exists m \in \text{people}) : m \text{ is mother of } p,$$

where people denotes the set of all people in the world (living or dead).

Then the assertion of this proposition is that every person possesses a mother, which is clearly true.

Now assume we exchange the order of the quantifiers involved to get the proposition

$$(\exists m \in \text{people})(\forall p \in \text{people}) : m \text{ is mother of } p.$$

Then this proposition asserts that there is one person in the world who is the mother of every person in the world (including herself), which is just as clearly false.