Lecture 10 — The Size of Infinite Sets

In this chapter we will consider different notions of infinity.

Definition 10.0.1. Let A be a set.

- The *cardinality* [Kardinalität] *$\#A$ of A is the number of elements of A.*
- *A is finite* [endlich] *if there is $n \in \mathbb{N}$ such that $\#A = n$. Otherwise A is infinite* [unendlich]*.*

Example 10.0.2.

- $\#\{1, 2, 3, 4, 5, \ldots, n\} = n$.
- $\#\{\text{apple, orange, pear, grape}\} = 4$.

Lemma 10.0.3. *$\#A = \#B$ iff there is a bijection $A \to B$.*

Example 10.0.4. Let $A = \{2, 3, 5, 7\}$ and $B = \{\text{apple, orange, pear, grape}\}$. Then $\#A = \#B$ and

$$\begin{aligned} f : A &\to B : \\ 2 &\mapsto \text{apple} \\ 3 &\mapsto \text{orange} \\ 5 &\mapsto \text{pear} \\ 7 &\mapsto \text{grape} \end{aligned}$$

is a bijection.

Recall that in Chapter 1 we introduced the natural numbers $\mathbb{N}$ as numbers that count objects.

Definition 10.0.5. A set A is *countable* [abzählbar] *if it has the same cardinality as (some subset of) $\mathbb{N}$ if there is an injective function $A \to \mathbb{N}$. Otherwise A is called uncountable* [überabzählbar]*.*

Note that every finite set is countable. Moreover, every infinite countable set has the same cardinality as $\mathbb{N}$.

10.1 Countable sets

Theorem 10.1.1. *Let A be a set. Then the following statements are equivalent:*

(i) *A is countable, i.e., there exists an injective function $A \to \mathbb{N}$.*

(ii) *Either A is empty or there exists a surjective function $\mathbb{N} \to A$.*

(iii) *Either A is finite or there exists a bijection $\mathbb{N} \to A$.*

Theorem 10.1.2. *Every subset of a countable set is countable.*

Theorem 10.1.3. *Let E denote the set $\{0, 2, 4, 6, \dots\}$ of even numbers. Then $\#E = \#\mathbb{N}$, i.e., there are equally many natural numbers and even numbers.*

Proof. We will prove this by giving a bijection $\mathbb{N} \to E$: The function $d : \mathbb{N} \to E : n \mapsto 2n$ is a bijection. □

Theorem 10.1.4. *$\mathbb{Z}$ is countable, i.e., $\#\mathbb{Z} = \#\mathbb{N}$.*

Proof. The function

$$f : \mathbb{N} \to \mathbb{Z} : n \mapsto \begin{cases} 0 & \text{if } n = 0, \\ \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

is a bijection. □

Exercise 10.1.5. The union of two countable sets is countable.

Exercise 10.1.6. The product of two countable sets is countable.

Cantor's first diagonal process

Theorem 10.1.7. *$\mathbb{Q}$ is countable, i.e., $\#\mathbb{Q} = \#\mathbb{N}$.*

Proof. We will prove this by exhibiting a surjective function $f : \mathbb{N} \to \mathbb{Q}$. We will first give a surjective function $f_+ : \mathbb{N} \to \mathbb{Q}_{\geq 0}$. Recall that we can represent every non-negative rational number q by two natural numbers a and b, $b \neq 0$ as $q = \frac{a}{b}$.

Write n as $n = \frac{k(k+1)}{2} + \ell$ such that $k, \ell \in \mathbb{N}$ and $0 \leq \ell \leq k + 1$. Now set $a := k - \ell$, $b = \ell + 1$. Consider the function

$$f_+ : \mathbb{N} \to \mathbb{Q}_{\geq 0} : n \mapsto \frac{a}{b}.$$

See Figure 10.1 for an illustration of this function. It is easy to see that f_+ is surjective.

Now define the function

$$f : \mathbb{N} \to \mathbb{Q} : n \mapsto \begin{cases} 0 & \text{if } n = 0, \\ f_+\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ -f_+\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

This is a surjective function $\mathbb{N} \to \mathbb{Q}$. □

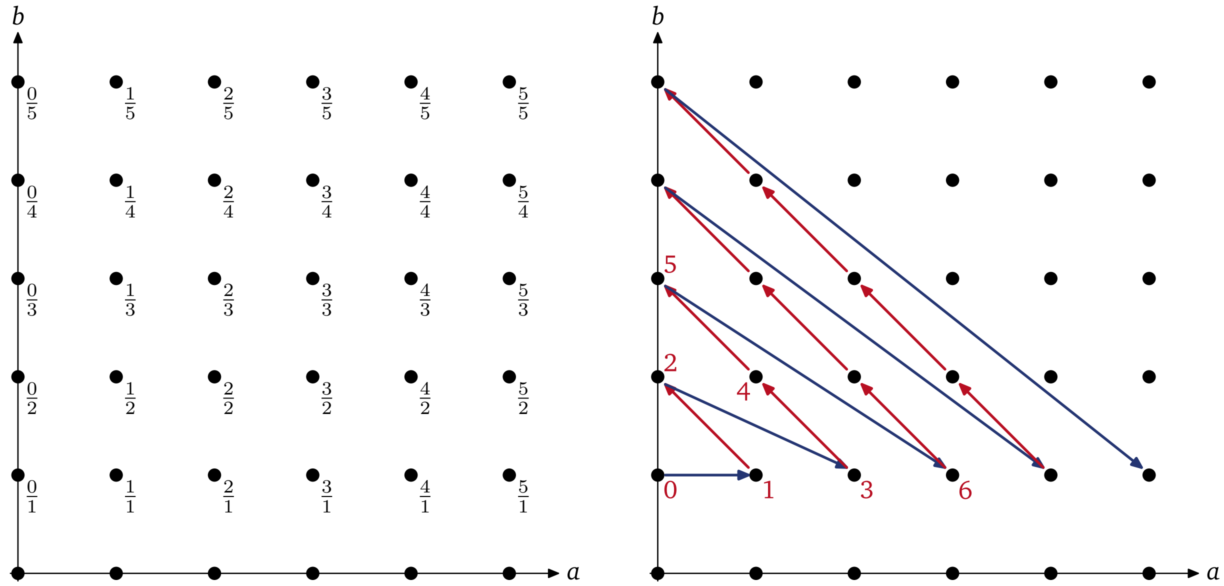

Figure 10.1: The non-negative rational numbers as pairs of natural numbers $a, b, b \neq 0$ on the left; the function $f_+ : \mathbb{N} \to \mathbb{Q}_{\geq 0}$ on the right. We see that f_+ is surjective.

10.2 Uncountable sets

Cantor's second diagonal process

Theorem 10.2.1. *$\mathbb{R}$ is uncountable.*

Proof. We will show that the interval $[0, 1]$ in $\mathbb{R}$ is uncountable. We will prove this by contradiction. Recall that every real number can be represented as an infinite sequence of digits.

So suppose that

$$\begin{array}{cccccc} s_{11} & s_{12} & s_{13} & \cdots & s_{1n} & \cdots \\ s_{21} & s_{22} & s_{23} & \cdots & s_{2n} & \cdots \\ s_{31} & s_{32} & s_{33} & \cdots & s_{3n} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ s_{m1} & s_{m2} & s_{m3} & \cdots & s_{mn} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{array}$$

is a list containing all real numbers in $[0, 1]$. Now consider the number $s = t_{11}\, t_{22}\, t_{33}\, \ldots\, t_{nn}\, \ldots$, where $t_{ii} \neq s_{ii}$. Then s is not contained in the list. □

This means that while there are equally many rational numbers as there are integers and natural numbers (although $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$), the set $\mathbb{R}$ of real numbers is strictly larger than $\mathbb{N}$.