Introductory Course Mathematics

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Title Picture by Prof. Dr. K. H. Hofmann

Introduction

These lecture notes are for the introductory course for beginning students of mathematics at TU Darmstadt.

The goal of this introductory course is to bring all students to a common level of knowledge, to repeat some of the more important contents you probably learned in school and to give you a first impression about what studying math is like. Moreover, since the course language for this course is English, the course aims to make you familiar with the English terminology. And because of all this, not only freshmen of the bilingual study programmes can benefit from this course but everyone else is also cordially invited to participate. (Even if one does not plan to do a bilingual programme, it might still be a good idea to gain some familiarity with the English vocabulary.) Therefore, I will, in addition to the English terms, also give German translations for most of the expressions. They will appear in brackets after the English term; for example *[Beispiel].*

Chapters 1 to 9 of these notes are based on the lecture notes originally written by Dr (AUS) Werner Nickel for the MCS introductory course and some modifications applied by Max Horn and Dennis Frisch.

If you find any mistakes in this text or have other suggestions, comments, remarks, questions, . . . of any sort, please feel free to contact me under:

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Finally, I would like to thank Prof. Dr. K. H. Hofmann for giving permission to use his wonderful drawing of the old main building on the title page of these lecture notes.

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Lecture 1 — Numbers

1.1 The Natural Numbers

The natural numbers [natürliche Zahlen] count objects, e.g. 3 *eggs,* 160 *students, about* 10^{70} *atoms in the universe.*

The set of natural numbers is denoted by $\mathbb{N}$. *Two natural numbers can be added and multiplied:*

$$3+5=8 \qquad 12\cdot 11=132 \qquad 7^2=49.$$

There are many interesting subsets of $\mathbb{N}$*, three of them are*

$2,4,6,8,10,\ldots$	*the even numbers [gerade Zahlen]*
$1,3,5,7,9,11,\ldots$	*the odd numbers [ungerade Zahlen]*
$1,4,9,16,25,\ldots$	*the perfect squares [Quadratzahlen].*

Any two natural numbers can be compared and they are either equal or one is smaller than the other. I.e., for any two natural numbers m *and* n *we have that*

$m<n$	*m is less than n*	*[m ist kleiner als n],*	
$m>n$	*m is greater than n*	*[m ist grösser als n],*	*or*
$m=n$	*m is equal to n*	*[m ist gleich n].*	

We thus say that the natural numbers are equipped with a total order [totale Ordnung].

Furthermore, if $a<b$ *and* $m<n$*, then* $a+m<b+n$ *and* $a\cdot m<b\cdot n$.

The elements of any subset of $\mathbb{N}$ *can be put in increasing order starting with the smallest element. Each non-empty subset of* $\mathbb{N}$ *has a unique smallest element.*

However, subsets of $\mathbb{N}$ *need not have a largest element.*

Exercise 1.1.1. Find an example of a subset of $\mathbb{N}$ that does not have a largest element. Describe all subsets of $\mathbb{N}$ that do have a largest element!

Definition 1.1.2. We say that a natural number n is *divisible [teilbar] by a natural number* d *if there exists a natural number* m *in* $\mathbb{N}$ *such that*

$$d\cdot m=n.$$

If this is the case, we also say that d *divides [teilt]* n *and write* $d|n$. *A natural number* d *that divides* n *is also called a divisor [Teiler] of* n. *Vice versa,* n *is a multiple [Vielfaches] of* d.

Example 1.1.3.

- The number 12 is divisible by 4.

  Proof: We need to use the definition above. Here we have that $n=12$ and $d=4$. We have to find a natural number m such that $12=4\cdot m$. This is easy since $m=3$ is such a number (in fact the only one).

- The number 12 is not divisible by 7.

  Proof: We need to show that there is no number m such that $7 \cdot m = 12$. (If there was such an m, then 7 would divide 12). In other words, we need to show that no multiple of 7 is equal to 12. The first few multiples of 7 are 7, 14, 21 which shows that 12 is not a multiple of 7.

Exercise 1.1.4.

- Prove: If d is a divisor of n, then $d = n$ or $d < n$.
- List all divisors of 12, 140 and 1001. Prove for 12 that there are no other divisors.
- Show that 7 is not a divisor of 100.
- Show that each natural number n is divisible by 1 and by n.
- Prove: If d divides m and n, then d also divides $m + n$ and $m - n$ and d^2 divides mn.

1.1.1 Prime numbers

Definition 1.1.5. A natural number different from 1 that is divisible by 1 and itself only is called a *prime number [Primzahl]*.

Examples of primes are: 2, 3, 5, 7, 2003, $2^{13} - 1$.

The definition of primes raises the question how one can find primes. This is a difficult problem in general. There is an algorithm which, in principle, can find all the primes, although it is impractical for large prime numbers.

The sieve of Eratosthenes

This procedure finds all primes up to a given bound. It works as follows: Choose a number N*, e.g.* $N = 20$*. List the natural numbers up to* N *beginning with* 2*:*

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

We iterate the following procedure: The next number which is not crossed out is a prime. We record it and cross out all its multiples:

So, 2 *is a prime. Cross out its multiples, the even numbers:*

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 ~~12~~ 13 ~~14~~ 15 ~~16~~ 17 ~~18~~ 19 ~~20~~

The next prime is 3*. Cross out all multiples of* 3*:*

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~.

The next prime is 5*. At this point we notice that all multiples of* 5 *have already been crossed out. The same is true for* 7 *and all the remaining numbers. Therefore, all the remaining numbers are primes.*

Trial division

How does one check if a natural number n is a prime? One way is to try if it is divisible by any smaller number. To do that one has to carry out $n-2$ divisions if n is a prime.

The following theorem helps in reducing the number of trial divisions because it shows that one only has to do trial divisions with smaller prime numbers. We have already used this fact in Eratosthenes' Sieve because we have declared a number a prime if it was not a multiple of any smaller prime.

Theorem 1.1.6. *Any natural number n is divisible by a prime.*

Proof. Consider all divisors of n different from 1. There is a smallest element q among these. Let m be a natural number such that $q \cdot m = n$.

We will show that q is a prime. Suppose that q is not a prime. Then q has a divisor $1 < d < q$ and $q = d \cdot m'$. We get

$$n = q \cdot m = (d \cdot m') \cdot m = d \cdot (m' \cdot m).$$

We see that d is a divisor of n. But d is smaller than q, which contradicts the choice of q. Therefore, it is impossible that q has a proper divisor. Hence, q is a prime. □

Theorem 1.1.7 (without a proof)**.** *Each natural number is a product of primes. This product is unique up to permuting the factors.*

Theorem 1.1.8. *There are infinitely many primes.*

Proof. We assume that there are only finitely many primes and show that this assumption leads to a contradiction.

Let k be the number of primes and let $p_1, p_2, p_3, \ldots, p_k$ be the finitely many primes. Consider $M := p_1 p_2 p_3 \ldots p_k + 1$. Clearly, p_j divides $p_1 p_2 p_3 \ldots p_k$. If p_j divides M, then p_j also divides $M - p_1 p_2 p_3 \ldots p_k = 1$. But no prime is a divisor of 1. Therefore, M is not divisible by any of the k primes above. This contradicts Theorem 1.1.6. □

We will now leave the prime numbers and turn our attention back to using natural numbers for counting.

1.1.2 Counting

Example 1.1.9. Consider the five vowels A E I O U. Here are all three-letter arrangements (without repitition of letters) of these:

AEI AEO AEU AIE AIO AIU AOE AOI AOU AUE AUI AUO EAI EAO EAU
EIA EIO EIU EOA EOI EOU EUA EUI EUO IAE IAO IAU IEA IEO IEU
IOA IOE IOU IUA IUE IUO OAE OAI OAU OEA OEI OEU OIA OIE OIU
OUA OUE OUI UAE UAI UAO UEA UEI UEO UIA UIE UIO UOA UOE UOI

If we want to write down all three-letter words, then we have 5 choices for the first letter. Once the first letter is fixed we have 4 choices for the second letter and after that 3 choices for the last letter. This gives $5 \cdot 4 \cdot 3 = 60$ different choices, each of which produces a different word.

The general argument goes like this: For the first object we have n choices. For the second object we have $n-1$ choices. As each choice of the first object can be combined with each choice of the second object, this gives $n(n-1)$ possibilities. For the third choice we have $n-2$ possibilities. Therefore there are $n(n-1)(n-2)$ possibilities to place 3 objects out of n objects in a row. In general, there are $n(n-1)(n-2)\dots(n-(k-1))$ possibilities to place k out of n objects in a row.

If $k=n$, then this gives $n(n-1)(n-2)\dots 3\cdot 2\cdot 1$ possibilities to arrange n different objects in a row. We denote the number

$$n(n-1)(n-2)\dots 3\cdot 2\cdot 1 =: n!,$$

which is pronounced *n factorial [n Fakultät]. We also set* $0! := 1$. *This can be interpreted as saying that there is one way to arrange no objects.*

Example 1.1.10. The four symbols $+ - \cdot /$ can be arranged in $24 = 4\cdot 3\cdot 2\cdot 1$ ways:

$$+-\cdot/ \quad +-/\cdot \quad +\cdot-/ \quad +\cdot/- \quad +/-\cdot \quad +/\cdot- \quad -+\cdot/ \quad -+/\cdot$$

$$-\cdot+/ \quad -\cdot/+ \quad -/+\cdot \quad -/\cdot+ \quad \cdot+-/ \quad \cdot+/- \quad \cdot-+/ \quad \cdot-/+$$

$$\cdot/+- \quad \cdot/-+ \quad /+-\cdot \quad /+\cdot- \quad /-+\cdot \quad /-\cdot+ \quad /\cdot+- \quad /\cdot-+$$

With the factorial notation we can write the number $n(n-1)(n-2)\dots(n-(k-1))$ as

$$\frac{n!}{(n-k)!}.$$

This counts the number of arrangements of k objects out of n objects.

If we were intersted in the number of ways there are to choose k objects out of n objects, then the order in which objects are chosen would be unimportant. The words AEI and EIA consist of the same letters and would not be considered different choices of three vowels.

Example 1.1.11. There are 10 ways to choose 3 vowels from A E I O U:

AEI AEO AEU AIO AIU AOU EIO EIU EOU IOU

In general, we need to take the number $\frac{n!}{(n-k)!}$ and divide by the number of arrangements of k objects. This gives

$$\frac{n!}{(n-k)!k!}.$$

This expression is abbreviated by

$$\frac{n!}{(n-k)!k!} =: \binom{n}{k},$$

which is pronounced as *n choose k [n über k]. Note that* $\binom{n}{0} = 1$. *This means that there is one way to choose no object out of n.*

Exercise 1.1.12.

- Show that $\binom{n}{k} = \binom{n}{n-k}$.

- Show that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

The last property can be used to compute these numbers in a systematic way, known as Pascal's Triangle [Pascalsches Dreieck]:

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
 & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
 & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\
 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & \\
 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & \\
 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & \\
1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1
\end{array}
$$

Each number is the sum of the two numbers above. The number $\binom{n}{k}$ is the k-th element in row n (counted from top to bottom), where we start counting from 0.

The numbers $\binom{n}{k}$ are called binomial coefficients [Binomnialkoeffizient]. The reason for this name becomes clear from the following: Consider the powers of the expression $x + y$. The first few are:

n	$(x+y)^n$
1	$x+y$
2	$x^2+2xy+y^2$
3	$x^3+3x^2y+3xy^2+y^3$
4	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$

Notation 1.1.13 (The Sigma Sign)**.** The following notation is a useful shorthand to concisely write sums and ubiquitous throughout mathematics.

Let's do an example: Consider the sum $1^2 + 2^2 + 3^2 + \ldots + k^2$. Here we simply used the dots and hoped that everyone would guess correctly what we mean by them. This is where the sigma notation comes in:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 =: \sum_{i=1}^{k} i^2.$$

The expressions below and above the sigma sign specify the index variable (i in that case), and all the values that i takes in the expression behind the sigma sign (all numbers between 1 and k).

Comparing the numbers in the expressions above with the numbers in Pascal's Triangle reveals the following connection:

Theorem 1.1.14 (*Binomial Theorem [Binomischer Lehrsatz]*)**.** *For each natural number n we have the following equality:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

The expression on the right hand side is the abbreviation of the sum

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{(n-1)} + \binom{n}{n}y^n.$$

We will give two proofs of this theorem:

Proof 1. We consider the coefficient of the expression $x^{n-k}y^k$ and how it arises from the product

$$\underbrace{(x+y)(x+y)\ldots(x+y)(x+y)}_{n\text{-times}}.$$

Expanding the brackets, we have to multiply each occurrence of x or y with each other occurrence of x or y. To obtain $x^{n-k}y^k$ we have to choose y exactly k-times. Since we are choosing k times y out of n occurrences of y, we can do this in $\binom{n}{k}$ ways. Therefore, the term $x^{n-k}y^k$ occurs $\binom{n}{k}$ times. □

Proof 2. We do a proof by *induction [Induktion].*

First, we show that the theorem is true for $n = 1$*:*

$$(x+y)^1 = x + y = \binom{1}{0}x + \binom{1}{1}y.$$

Now we show that the statement of the theorem is true for $n+1$ *if it is true for a natural number* n*.*

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\cdot\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k+1}y^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=1}^{n+1}\binom{n}{k-1}x^{n+1-k}y^k \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)x^{n+1-k}y^k + y^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k
\end{aligned}$$

□

1.2 The Integers

The set $\mathbb{N}$ is closed [abgeschlossen] under taking sums and products of natural numbers. I.e., sums and products of natural are again natural numbers.

However the difference of two natural numbers need not be a natural number: $7 - 13 =?$ In other words, there is no solution to the equation $7 = x + 13$ in the set of natural numbers.

Thus, one defines the set $\mathbb{Z}$ of integers [ganze Zahlen]:

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

The integers are closed under taking sums, products and differences.

The notions of order, divisibility and primes defined above can be extended to the set of integers in a natural way with little changes:

Theorem 1.2.1 (without proof)**.** *Each integer is a product of primes and ± 1. This product is unique up to permuting the factors.*

1.3 The Rational Numbers

The quotient of two integers need not be an integer. In fact, the quotient of an integer m and an integer d is an integer if and only if m is divisible by d. In other words, for integers m and n the equation $m \cdot x = n$ need not have a solution for x in the set of integers.

Again, we define a new set to sort out this problem, the set $\mathbb{Q}$ of rational numbers [rationale Zahlen]. It consists of all fractions [Brüche] $\frac{a}{b}$ where a is an integer and b a non-zero integer. The integer a is called the numerator [Zähler] and b is called the denominator [Nenner]. The rationals are closed under addition, subtraction, multiplication and division.

A set of numbers in which those four arithmetic operations can be performed is called a field [Körper]; $\mathbb{Q}$ is called the field of rationals numbers [Körper der rationalen Zahlen].

Arithmetic of rational numbers

Definition 1.3.1. *Let a, b, c and d be integers with b and d not 0.*

- ***Addition*** *Two fractions are added by finding a common denominator (you may want to look for their smallest common denominator):*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

- ***Multiplication*** *Two fractions are multiplied by multiplying numerators and denominators:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

- ***Division*** *A fraction is divided by another fraction by multiplying with the* reciprocal *[Kehrwert] of the second fraction ($c \neq 0$):*

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

- *Between any two different rational numbers lie infinitely many rational numbers. For this it is enough to show that there is always a rational number lying strictly between any pair of different rational numbers. For example, a rational number lying between the rational numbers x and y is the number $\frac{x+y}{2}$.*

- ***Equality*** *Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $ad = bc$.*

This definition implies that canceling common factors in the numerator and denominator of a fraction does not change the value of the fraction: Let a, b and c be integers with b and c different from 0. Then

$$\frac{ac}{bc} = \frac{a}{b} \quad \text{because} \quad ac \cdot b = bc \cdot a.$$

However, certain equations do not have a solutions in the set of rational numbers. For example, the equation $x^2 = 2$.

Theorem 1.3.2. *A solution of the equation $x^2 = 2$ is not a rational number.*

Proof. Let $\frac{a}{b}$ be a rational number with $\left(\frac{a}{b}\right)^2 = 2$. We may assume that a and b have no common factor.
Then $a^2 = 2b^2$. Therefore a^2 is an even number. The square of an integer is even if and only if the integer is even. Therefore, a is even and can be written as $a = 2d$. This gives $2b^2 = 4d^2$ and dividing by two gives $b^2 = 2d^2$. By the same reasoning as above, b is even. Hence a and b contain the common factor 2, contrary to our assumption. □

Exercise 1.3.3.

(i) Let n be a natural number. Show that n^2 is even if and only if n is even.

(ii) Show that $x^2 = 6$ does not have a rational solution.

(iii) Show that $1 + \sqrt{2}$ is not a rational number.

(iv) Show that $x^3 = 2$ does not have a rational solution.

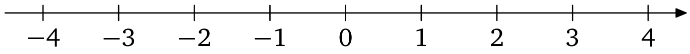
1.4 The Real Numbers

The set of real numbers [reelle Zahlen], denoted by $\mathbb{R}$ is an extension of the rational numbers containing all limits [Grenzwerte] of rational sequences [Folgen] such as

$$\begin{aligned}
\sqrt{2} &= 1{,}4142135623730950488016887242096980785696718753769480731766797379907324784621\ldots \\
e &= 2.7182818284590452353602874713526624977572470936999595749669676277240766303535\ldots \\
\pi &= 3.1415926535897932384626433832795028841971693993751058209749445923078164062862\ldots
\end{aligned}$$

and the solutions to equations of the form $x^5 + x + 1 = 0$ and many more. The real numbers are much more complicated than the rational numbers. Most real numbers cannot be written down explicitly.

However, one important feature of the reals is that they – just as the rationals – form a field. The set of real numbers is often visualised by a line, called the real line [reelle Zahlengerade].



Definition 1.4.1. Let a be a non-negative real number and n a natural number. The *n-th root [n-te Wurzel] of a is a non-negative real number r such that $r^n = a$.*

Note that in general the n-th root is only defined for non-negative real numbers. Also the n-th root of a non-negative real number is always a non-negative real number. Taking the root of a positive number is the inverse operation to raising a real number to the n-th power.

If x is a negative number, then taking the square root is not the inverse operation of squaring x because the square root is positive: $x \neq \sqrt{x^2} = -x$. The same is true for all even powers n. If n is an odd number, however, then the n-th root is declared for all real numbers x, e.g.

$$\sqrt[3]{-8} = -2.$$

Definition 1.4.2. Let a be a real number. We define the following function:

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0. \end{cases}$$

The non-negative real number $|a|$ is called the *absolute value [Betrag] of a.*

Exercise 1.4.3. Let a, b and c be real numbers and ε a positive real number.

- Show that $|a| \leq c$ is the same as saying $-c \leq a \leq c$.
- Show that $a \leq |a|$ and $-|a| \leq a$.
- Prove the triangle inequality: $|a + b| \leq |a| + |b|$. *Hint*: Use the previous two inequalities.
- Prove the inequality $|a| - |b| \leq |a - b|$.
- Show that $|x - a| \leq \varepsilon$ is the same as saying $a - \varepsilon \leq x \leq a + \varepsilon$. Interpret this geometrically! What is the set of all x satisfying this condition?
- Determine the solutions of the inequalities $|4 - 3x| > 2x + 10$ and $|2x - 10| \leq x$.

1.5 The Complex Numbers

The real numbers allow us to solve many more equations than the rational numbers, which in turn allow solving more equations than the integers. Still, there are some simple equations we cannot solve. In particular, the equation $x^2 + 1 = 0$ has no solution over the reals. A solution to this would be $\sqrt{-1}$ if it were defined.

When faced with the problem of not being able to divide by arbitrary non-zero numbers, we simply introduced new symbols (namely fractions). We do the same with the square root of -1 by defining the symbol i (the imaginary unit) such that

$$i^2 = -1.$$

This leads to the set $\mathbb{C}$ of complex numbers [komplexe Zahlen]. It consists of all terms of the form $z = a + bi$, where a and b are real numbers. We call a the real part [Realteil], and b the imaginary part [Imaginär Teil] of z. The complex numbers form a field with the real numbers naturally embedded in them. Unlike the number sets we saw so far, the complex numbers do not permit a natural total order.

Arithmetic of complex numbers

Definition 1.5.1. Let $a, b, c, d \in \mathbb{R}$.

- **Equality** Two complex numbers $a + bi$ and $c + di$ are equal if and only if their real and imaginary parts are equal, *i.e.*, if $a = c$ and $b = d$.
- **Addition** Two complex numbers are added as one might expect:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

- **Multiplication** Two complex numbers are multiplied by following the normal rules of multiplication, treating i like a variable and using that $i^2 = -1$:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

- **Division** A complex number is divided by another (non-zero) complex number by multiplying by the *inverse [Inverse] of the second number. The inverse is computed as follows:*

$$(a + bi)^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i.$$

Exercise 1.5.2. Verify that the inversion formula in Definition 1.5.1 is correct.

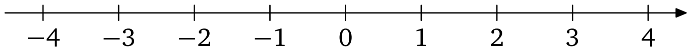
We define the complex conjugate [komplex konjugierte] of the complex number $z = a + bi$ as $\bar{z} := a - bi$. We now define the absolute value for a complex number z as

$$|z| := \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

Note that over the real numbers this coincides with the previous definition of absolute value. Using these notations, we can write z^{-1} as $\frac{\bar{z}}{|z|^2}$.

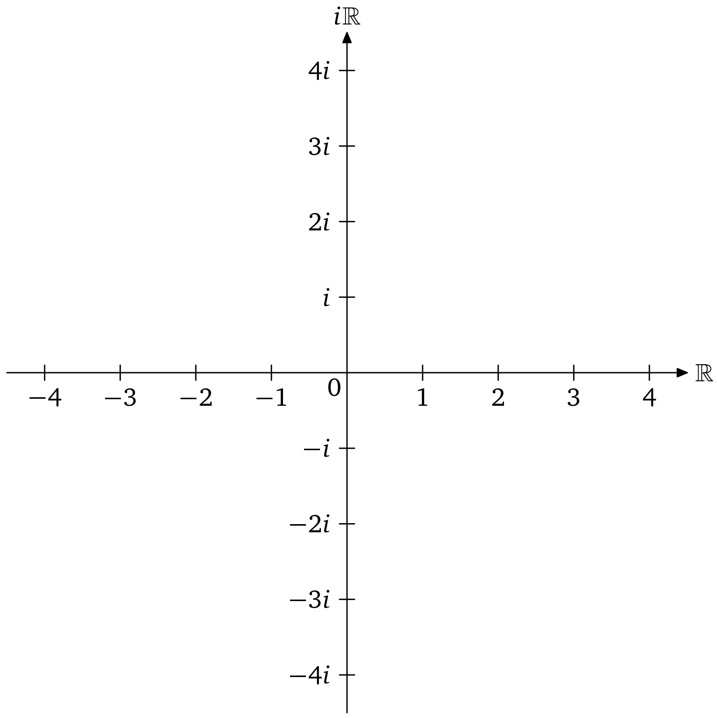
Complex numbers from a geometric point of view

When introducing the real numbers, we also introduced the real line.



The real line is a geometric way to visualise the real numbers. So let's try and find out how the arithmetic operations are reflected in this geometric setting. We can easily see that additon is a translation and multiplication is a dilation. If the number we multiply by is negative, then the dilation also changes the direction.

We can view the complex numbers as a product of two real lines, one for the real part and one for the imaginary part. We can visualise complex numbers in a coordinate system:



Now we can view the field of complex numbers as a two dimensional plane, the so called complex plane *[komplexe Zahlenebene].*

There is another possibility to describe complex numbers:

Proposition 1.5.3. *Each complex number $z = a + bi$ can be expressed as*

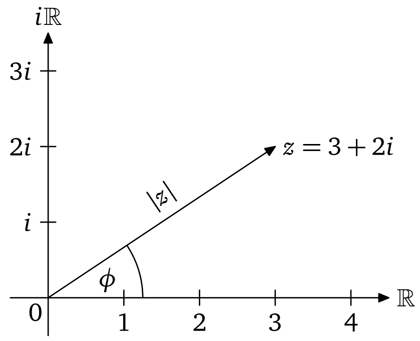
$$z = |z|(\cos\phi + i\sin\phi),$$

where ϕ is a real number called the argument *[Argument] of z and*

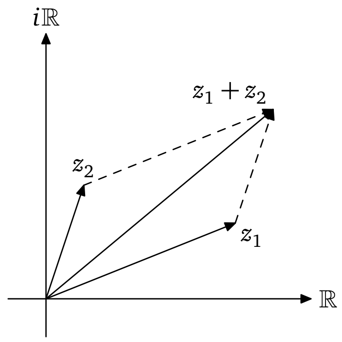
$$|z| := \sqrt{a^2 + b^2}$$

is the absolute value of z. If we take $-\pi < \phi \leq \pi$, then ϕ is uniquely determined.

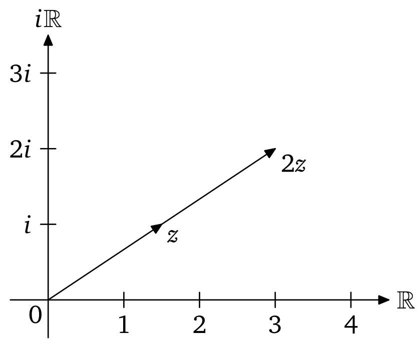
Moreover, if we draw z as a vector in the complex plane, then $|z|$ is its length and ϕ is the angle between the vector and the real axis.



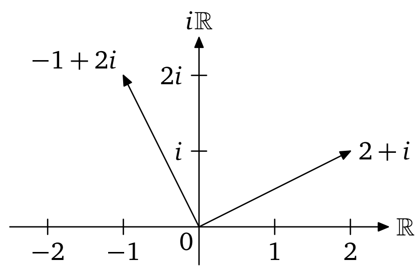
We know from above that adding a real number to a real number is a translation. This is still true for complex numbers.



Looking at the multiplication, we saw that multiplying a real number by a real number is a dilation.
This is still correct, if we multiply a complex number by a real number.



Now consider a complex number with non-zero imaginary part:
Let $z := 2 + i$. *If we multiply by* i *we get* $(2+i) \cdot i = 2i + i^2 = -1 + 2i$.
We see that $|2+i| = |-1+2i|$ *and in the complex plane we see that a multiplication by* i *results in a rotation about 90 degrees counterclockwise.*



In general, if we multiply two complex numbers z_1 and z_2, we multiply the lengths and add the angles.

Proposition 1.5.4. *Let* $z_1 = \left|z_1\right| (\cos\phi_1 + i \sin\phi_1)$, $z_2 = \left|z_2\right| (\cos\phi_2 + i\sin\phi_2)$, *then*

$$z_1 \cdot z_2 = \left|z_1\right| \left|z_2\right| (\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)).$$

Lecture 2 — Propositional Logic

Examples of propositions: 5 *is not a number. Darmstadt is in Germany. Mathematics is a science.* 7 *divides* 12*.*

A proposition [Aussage] is a grammatically correct statement which it can be decided of whether it is true or false.

More interesting than deciding whether one proposition is true or false is to decide whether a proposition is true under certain circumstances. This process is fundamental in mathematics.

We now have a look at how to combine given propositions to new propositions and under which circumstances the new proposition is true.

2.1 Logical operators

Whenever we construct a new proposition from other propositions we can use a truth table [Wahrheitstabelle] to describe the newly constructed proposition. We simply write down the value of the new one for all combinations of values of the old ones. This section gives several examples for truth tables.

Negation: *The negation [Verneinung] of a proposition A is false when A is true and vice versa (written $\neg A$):*

A	$\neg A$
t	f
f	t

And (Conjunction): *Two propositions A and B can be combined by "and" to give a new proposition $A \wedge B$ (the conjunction [Konjunktion] of A and B) which is true precisely when both A and B are true:*

A	B	$A \wedge B$
t	t	t
t	f	f
f	t	f
f	f	f

Or (Disjunction): *Two propositions A and B can be combined by "or" to give a new proposition $A \vee B$ (the disjunction [Disjunktion] of A and B) which is true precisely when at least one of A and B is true:*

A	B	$A \vee B$
t	t	t
t	f	t
f	t	t
f	f	f

Implication: *If a proposition B is true whenever another proposition A is true, then A implies [impliziert] B. We write $A \Rightarrow B$ and call this statement an implication [Implikation]:*

A	B	$A \Rightarrow B$
t	t	t
t	f	f
f	t	t
f	f	t

Equivalence: *A proposition A is equivalent [äquivalent] to a proposition B (written $A \Leftrightarrow B$) if A is true precisely when B is true and A is false precisely when B is false (also written A iff B, which means A is true if and only if B is true).*

A	B	$A \Leftrightarrow B$
t	t	t
t	f	f
f	t	f
f	f	t

We give another characterisation for equivalence. And we take this as an example for a typical proof of such logical propositional statements:

Theorem 2.1.1. *Let A and B be two propositions. Then the following are equivalent:*

(i) $((A \Rightarrow B) \wedge (B \Rightarrow A))$

(ii) $(A \Leftrightarrow B)$

Proof.

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$(A \Rightarrow B) \wedge (B \Rightarrow A)$
t	t	t	t	t
t	f	f	t	f
f	t	t	f	f
f	f	t	t	t

□

Implications that are not equivalences

Here are some examples for implications which are only true "in one direction", i.e., they are no equivalences:

- *For all $x \in \mathbb{R}$: $x > 0 \Rightarrow x^2 > 0$.*
- *If x and y are negative real numbers, then $x \cdot y > 0$.*

Now let us have a look at the converses [Umkehrung] of the above propositions:

- *For all $x \in \mathbb{R}$: $x^2 > 0 \Rightarrow x > 0$.*
- *If $x \cdot y > 0$ then x and y are negative real numbers.*

In both cases we easily find a counterexample to refute these propositions.

2.2 Quantifiers

Another important feature of propositional logic are the quantifiers [Quantoren]:

All quantifier *[All-Quantor]: If for each element e of a set S a proposition $A(e)$ is given then*

$$\forall e \in S : A(e)$$

is a proposition which is true iff $A(e)$ is true for each $e \in S$. (Read: For all e in S, $A(e)$ is true.)

Existence quantifier *[Existenz-Quantor]: If for each element e of a set S a proposition $A(e)$ is given then*

$$\exists e \in S : A(e)$$

is a proposition which is true iff $A(e)$ is true for at least one $e \in S$. (Read: It exists an element e such that $A(e)$ is true.)

2.3 Negation of propositions

Consider the following examples:

- *All sheep are black.*
- *It exists a male student at the TU Darmstadt.*
- *An animal is a lion or a duck.*
- *A real number is positive and negative.*

Now consider the negations of the above:

- *If all sheep are black, then there is no sheep with another colour. So the negation is: There exists a sheep which is not black.*
- *This proposition is true if at least one student at the TU Darmstadt is male. So the negation is: All students at the TU Darmstadt are not male.*
- *To be true, each animal has to be a duck or a lion. So the negation is: There is an animal which is neither a duck nor a lion.*
- *The proposition is true if all real numbers are both positive and negative. So the negation is: There is a real number which is not negative or not positive.*

We summarise these insights using quantifier symbols:

Proposition 2.3.1.

- $\neg(\forall e \in S : A(e)) \iff \exists e \in S : \neg A(e)$
- $\neg(\exists e \in S : A(e)) \iff \forall e \in S : \neg A(e)$
- $\neg(A \vee B) \iff \neg A \wedge \neg B$
- $\neg(A \wedge B) \iff \neg A \vee \neg B$

As a rule of thumb you can keep in mind that a negation moving to and fro inside a proposition flips every quantifier it passes.

2.4 The order of quantifiers

It is very important to understand that we cannot change the order of the quantifiers in a proposition. E.g. let $A(x,y)$ *be a proposition defined for each* x *and* y*. Then the proposition* $(\forall x)(\exists y) : A(x,y)$ *is not equivalent to the proposition* $(\exists y)(\forall x) : A(x,y)$*.*

This is immediately evident if you consider the following everyday example:

Example 2.4.1. Consider the following proposition:

$$(\forall p \in \text{people})(\exists m \in \text{people}) : m \text{ is mother of } p,$$

where people denotes the set of all people in the world (living or dead).

Then the assertion of this proposition is that every person possesses a mother, which is clearly true.

Now assume we exchange the order of the quantifiers involved to get the proposition

$$(\exists m \in \text{people})(\forall p \in \text{people}) : m \text{ is mother of } p.$$

Then this proposition asserts that there is one person in the world who is the mother of every person in the world (including herself), which is just as clearly false.

Lecture 3 — Proof Techniques

A proof is a convincing demonstration (within the accepted standards of the field) that some mathematical statement is necessarily true. Proofs are obtained from deductive reasoning, rather than from inductive or empirical arguments. That is, a proof must demonstrate that a statement is true in all cases, without a single exception.

In this chapter we will take a look at the most common and expedient proof techniques and give examples to illustrate them.

3.1 Direct proof

A direct proof [direkter Beweis] is straightforward. The conclusion is established by logically combining the axioms, definitions, and earlier theorems. Thus, you start out with the assumptions A and try to directly conclude proposition B. I.e., we want to show $A \implies B$.

Example 3.1.1.

- The sum of two even integers is even.
- If a divides b and a divides c then a divides $b + c$.

3.2 Proof by contradiction

A proof by contradiction [Beweis durch Widerspruch] (also known as reductio ad absurdum) uses the fact that a proposition can only be either true or false.

It is shown that if some statement (namely, the negation of what we aim to prove) were true, then a logical contradiction would have to occur.

Example 3.2.1.

- $\sqrt{3}$ is irrational.
- There is no smallest positive (*i.e.*, > 0) rational number.

3.3 Proof by contraposition

A proof by contraposition [Beweis durch Kontraposition] uses

$$(A \implies B) \iff (\neg B \implies \neg A).$$

In other words, it establishes the conclusion "if A then B" by proving the equivalent contrapositive statement "if not B then not A".

Example 3.3.1. If n^2 is even, then n is even.

3.4 Proof by induction

A proof by induction [Beweis durch vollständige Induktion] is a useful tool to prove a proposition $B(n)$ which is stated for all natural numbers n (or for all natural numbers greater than a given number $k \in \mathbb{N}$).

A proof by induction consists of two parts:

(i) *Induction beginning [Induktionsanfang]: Prove $B(1)$ (or $B(k)$).*

(ii) *Induction step [Induktionsschritt]: Assume that $B(n)$ is true and show that $B(n+1)$ is true (making use of the fact that $B(n)$ is true).*

Imagine a domino chain. The induction step assures that a domino (here $n+1$) falls if domino n falls.

Look at this example:

Example 3.4.1. A typical example for a proof by induction is the following:

Proposition: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proof. We do a proof by induction.

Induction beginning: For $n = 1$, the claim is

$$\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2},$$

which is quite obviously true.

Induction step: Now we assume that the proposition is true for n. That means

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

is true. With the help of this assumption we try to prove the proposition

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2},$$

which is the proposition for $n+1$. We get

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\
&\overset{\text{assumption}}{=} \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}.
\end{aligned}$$

□

To see that both steps (beginning and induction) are essentiall for the proof to be valid, consider the following:

Example 3.4.2. Consider the proposition: $1+n > 2+n$ for all $n \in \mathbb{N}$.
This is obviously wrong, but still it is easy to do the induction step:
Assume that the proposition is true for n. So

$$\begin{aligned} & n+1 > n+2 \\ \iff & n+1+1 > n+2+1 \\ \iff & (n+1)+1 > (n+1)+2. \end{aligned}$$

But we cannot do the induction beginning because there is no smallest natural number satisfying $n+1 > n+2$.

A fake induction proof

Some proofs look good at the first glance, but sometimes a subtle error is inside. Look at this "proof":

Theorem: *All sheep have the same colour.*

Proof. We proof inductively that any set of sheep consists of only sheep of a single colour, *i.e.*, is equicolored.
Induction start: A set containing one sheep is obviously equicolored.
Induction step: Assume that any set of n sheep is equicolored. Now consider a set of $n+1$ sheep. The set formed by the first n sheep is equicolored. But so is the set formed by the last n sheep. Hence the whole set must be equicolored. □

What is wrong here?

3.5 Proof strategies

In mathematics, you often state the existence of a certain object. For example:

Theorem 3.5.1. *Any two natural numbers a and b have a* greatest common divisor *[größter gemeinsamer Teiler].*

We will first consider two different proofs of Theorem 3.5.1.

Existence proof

Like the name promises, an existence proof *[Existenzbeweis] proves the existence of something. Let's look at Theorem 3.5.1:*

Proof. A divisor of a natural number has to be less or equal to the number. Since there are only finitely many natural numbers which are less than or equal to a and b, respectively, only finitely many divisors can exist.

Take 1: it is a natural number and divides both a and b. So 1 is a common divisor. Since there are only finitely many other divisors, a greatest common divisor exists. □

At the end of the proof, we know that the theorem holds but we cannot say what the greatest common divisor is.

Constructive proof

A constructive proof [konstruktiver Beweis] is a proof which also delivers a solution. Again we look at Theorem 3.5.1:

Proof. For each prime number p denote by $e_a(p), e_b(p)$ its exponent in the (unique!) prime factor decompositions of a, respectively b. (Note that in either case only finitely many of the $e_a(p), e_b(p)$ are larger than zero.)

Then

$$d := \prod_{p \text{ prime}} p^{\min(e_a(p), e_b(p))}$$

is the greatest common divisor of a and b. □

Uniqueness proof

Once we know about the existence of a solution, we might be interested in its uniqueness. Then we have to maintain a uniqueness proof [Eindeutigkeitsbeweis].

Consider the following theorem, which is stronger than Theorem 3.5.1.

Theorem 3.5.2. *Any two natural numbers a and b have a* unique *greatest common divisor.*

Proof. Assume that there are two different greatest common divisors c and d of a and b. Since c and d are common divisors and c is the greatest common divisor, we must have $c \geq d$. The same is true for d leading to $d \geq c$ and so $c = d$. □

3.6 The pigeon hole principle

The following – surprisingly simple – theorem can very often be applied in proofs:

Theorem 3.6.1 (The *pigeon hole principle [Schubfachprinzip]*)**.** *Assume we are given $n+1$ balls and n boxes and are asked to distribute the balls among the boxes. Then there exists one box which contains more than one ball.*

There is nothing to prove.

Now let us apply the pigeon hole principle to prove the following:

Proposition 3.6.2. *Let $n \in \mathbb{N}$ with $2 \nmid n$, $5 \nmid n$. Then there is a number $N \in \mathbb{N}$ whose decimal representation is $N = 111\ldots111$ such that $n | N$.*

Proof. Denote by $N_1, N_2, \ldots, N_{n+1}$ the first $n+1$ numbers of the form $111\ldots111$, *i.e.*,

$$N_i = \underbrace{111\ldots111}_{i \text{ ones}}.$$

Moreover, denote by R_i the remainder of N_i on division by n. Then $0 \leq R_i \leq n-1$. Thus, there are only n different possible values for R_i and by the pigeon hole principle (Theorem 3.6.1) there are $i \neq j$ such that $R_i = R_j$. Without loss of generality assume that $i > j$. Then $n|(N_i - N_j)$ and

$$N_i - N_j = \underbrace{111\ldots111}_{i-j \text{ many}}\underbrace{000\ldots000}_{j \text{ many}} = \underbrace{111\ldots111}_{i-j \text{ many}} \cdot 10^j = N_{i-j} \cdot 10^j.$$

Since n is relatively prime to 10^j, we must have $n|N_{i-j}$. □

Lecture 4 — Functions

Definition 4.0.3. Let X and Y be sets. A *function [Funktion] f from the set X to the set Y, denoted $f : X \to Y$, is a rule that assigns to each element of X exactly one element of Y.*

The element of Y assigned to a particular element $x \in X$ is denoted by $f(x)$ and is called the image of x under f [Bild von x unter f]. Vice versa, x is called a preimage [Urbild] of $y = f(x)$. Note that an element $y \in Y$ can have more than one preimage under f or may not a have a preimage at all.

The set X is called the domain [Definitionsbereich] of f and Y is called the range [Wertebereich] of f. The set $\{f(x) \mid x \in X\}$ of all images is called the image [Bild] of f.

It is important to understand that the domain and the range are an essential part of the definition of a function. For example, consider the functions

$$\begin{aligned} f :&\ \mathbb{R} \to \mathbb{R} && x \mapsto x^2 \\ g :&\ \mathbb{R} \to \mathbb{R}_{\geq 0} && x \mapsto x^2 \end{aligned}$$

Strictly speaking, these are two different functions. One obvious difference is that all elements in the range of g do have a preimage, while there are elements in the range of f which do not have a preimage (-1 for example). So the statement "All elements in the range have a preimage." is true for g and false for f.

Example 4.0.4.

(i) Let $c \in Y$ be constant. Then the function

$$\begin{aligned} f :\ X &\to Y \\ x &\mapsto c \end{aligned}$$

is called a *constant function [konstante Funktion]. It maps each element of X to the same value c.*

(ii) *The function*

$$\begin{aligned} \mathrm{id}_X :\ X &\to X \\ x &\mapsto x \end{aligned}$$

is called the identity function [Identität] of X. It maps each element of X to itself.

4.1 Properties of functions

Definition 4.1.1. Let $f : X \to Y$ be a function.

- The function f is called *injective [injektiv] iff for all $x_1, x_2 \in X$*

$$x_1 = x_2 \iff f(x_1) = f(x_2).$$

- *The function f is called surjective [surjektiv] iff for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.*
- *If f is injective and surjective then it is bijective [bijektiv], i.e., f is bijective iff for each $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$.*

Example 4.1.2.

(i) The function id_X is bijective.

(ii) The constant function $f : x \mapsto c$ for a fixed c is injective if and only if X has exactly one element. It is surjective if and only if Y has exactly one element.

(iii) The function

$$\begin{array}{rccc} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x(x-1)(x+1) \end{array}$$

is not injective because $f(-1) = f(0) = f(1) = 0$. The function is surjective because the equation $f(x) = c$ is equivalent to the equation $x^3 - x - c = 0$, which is a polynomial of degree three, which has a zero in $\mathbb{R}$.

4.2 Algebra with functions

Definition 4.2.1. We consider functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then we can construct new functions

(i) $f \pm g : x \mapsto f(x) \pm g(x)$ for $x \in X \cap Y$,

(ii) $f \cdot g : x \mapsto f(x) \cdot g(x)$ for $x \in X \cap Y$,

(iii) $\frac{f}{g} : x \mapsto \frac{f(x)}{g(x)}$ for $x \in X \cap Y$ and $g(x) \neq 0$,

(iv) $g \circ f : x \mapsto g(f(x))$ if $f(X)$ is contained in Y.

This is called the *composition [Hintereinanderausführung/Verkettung] of functions. The function f is the inner function [innere Funktion] and the function g is the outer function [äußere Funktion].*

Example 4.2.2. Consider the function $f : \mathbb{R} \to \mathbb{R} : f(x) = \sqrt{x^2+1}$ and decompose it as follows: Let $\mathbf{1}_{\mathbb{R}} : x \mapsto 1$ and $\sqrt{\cdot} : x \mapsto \sqrt{x}$. Then

$$f = \sqrt{\cdot} \circ (\mathrm{id}_{\mathbb{R}} \cdot \mathrm{id}_{\mathbb{R}} + \mathbf{1}_{\mathbb{R}})$$

Theorem 4.2.3. *Let $f : X \to Y$ be a bijective function. Then there is a unique function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.*

The function g is called the inverse function [Umkehrfunktion] of f. We write $g = f^{-1}$. If $f(x) = y$, then $f^{-1}(y) = x$.

4.3 Types of functions on $\mathbb{R}$

The following is a list of certain frequently appearing types of functions on $\mathbb{R}$.

constant functions *Let $c \in \mathbb{R}$. Then a function $f(x) = c$ is a constant function.*

power functions *The function $f(x) = x^n$ for a natural number n is called a power function [Potenzfunktion].*

polynomials *A function of the form $f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$ is called a polynomial function [Polynom]. Polynomial functions are built from the identity function $\mathrm{id}_{\mathbb{R}}$ and the constant functions using $+, -, \cdot$.*

rational functions *A function of the form $f(x) = p(x)/q(x)$ with polynomials p and q is called a rational function [rationale Funktion]. Note that its maximal domain is $\mathbb{R} \setminus \{x \in \mathbb{R} \mid q(x) = 0\}$.*

algebraic functions *Algebraic functions [algebraische Funtionen] are constructed from polynomials (or, equivalently from the identity function and the constant functions) by using $+$, $-$, $\cdot$, $/$ and taking roots.*

Lecture 5 — Sequences

5.1 Examples and definition

The mathematical concept of a sequence [Folge] is easy to understand. First we look at a few examples.

Example 5.1.1.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots \quad \text{the sequence of natural numbers}$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \quad \text{a sequence of rational numbers}$$

$$-1, 1, -1, 1, -1, 1, -1, \dots \quad \text{a sequence of 1s and } -1\text{s}$$

$$\pi, \frac{2}{3}, 15, \log 2, \sqrt{15}, \dots \quad \text{a sequence of random real number}$$

The characteristic feature of a sequence of numbers is the fact that there is a first term of the sequence, a second term, and so on. In other words, the numbers in a sequence come in a particular order. This gives rise to the following formal definition:

Definition 5.1.2. A sequence of real numbers is a map from the natural numbers $\mathbb{N}$ into the set $\mathbb{R}$ of real numbers. This means that for each natural number n there is an element of the sequence, which we denote by a_n. In this notation, the elements of the sequence can be listed as

$$a_1, a_2, a_3, \dots$$

More concisely, we write $(a_n)_{n\in\mathbb{N}}$ for the sequence.

Example 5.1.3.

(i) Let c be a fixed constant real number. Then the sequence $a_n = c$ for $n \in \mathbb{N}$ is called *constant sequence [konstante Folge].*

$$c, \quad c, \quad c, \quad c, \quad c, \quad c, \quad c, \quad \dots$$

(ii) $a_n = \frac{1}{n}$ *for* $n \in \mathbb{N}$. *The term* a_{17} *is* $\frac{1}{17}$. *A sequence like this is defined explicitly. It is given by a formula which can be used directly to compute an arbitrary term of the sequence.*

(iii) *Here is another example of an explicitly given sequence:* $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$.

(iv) *Define* $a_1 = 1$ *and* $a_{n+1} = a_n + (2n+1)$. *This is a recursively [rekursiv] defined sequence. To compute* a_{n+1} *we need to know* a_n, *for which we need to know* a_{n-1} *and so on. Sometimes it is not difficult to find an explicit description for a recursively defined sequence. In this case we have* $a_n = n^2$.

(v) *A famous (and more difficult) example for a recursively defined sequence is the Fibonacci sequence:* $f_1 = 1$, $f_2 = 1$ *and* $f_{n+1} = f_n + f_{n-1}$ *for* $n > 2$, $n \in \mathbb{N}$. *The first few terms of the sequence are*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

There is the following closed form:

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

5.2 Limits and convergence

We will now have a closer look at the terms of the sequences (ii) and (iii) from Example 5.1.3 above:

$$1, \quad \frac{1}{2} = 0.5, \quad \frac{1}{3} = 0.3333\ldots, \quad \frac{1}{4} = 0.25, \frac{1}{5} = 0.2, \quad \ldots, \quad \frac{1}{200} = 0.005, \quad \ldots$$

$$\frac{1}{2} = 0.5, \quad \frac{2}{3} = 0.6666\ldots, \quad \frac{3}{4} = 0.75, \quad \frac{4}{5} = 0.8, \quad \ldots, \quad \frac{199}{200} = 0.995, \quad \ldots$$

While the terms of the first sequence get closer and closer to 0*, the terms of the second sequence get closer and closer to* 1*. Although no term of either sequence ever reaches* 0 *or* 1*, respectively, we would like to be able to express the fact that both sequences approach a certain number and get arbitrarily close.*

Definition 5.2.1. A sequence $(a_n)_n \in \mathbb{N}$ has a *limit [Grenzwert]* $a \in \mathbb{R}$ *if for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$\left|a_n - a\right| < \varepsilon \text{ for } n \geq N.$$

If the sequence $(a_n)_{n\in\mathbb{N}}$ *has a limit* a*, then* $(a_n)_{n\in\mathbb{N}}$ *is called* convergent *[konvergent] and we write*

$$\lim_{n\to\infty} a_n = a.$$

Read: The limit of a_n *as* n *goes to* ∞ *is* a*.*

If a sequence is not convergent it is called divergent *[divergent].*

It is worthwhile to think about this definition for a while and understand what the different parts of the definition mean. One way to interpret it is to say that a *is a limit of a sequence* $(a_n)_{n\in\mathbb{N}}$ *if the distance of* a *to all except a finite number of terms of the sequence is smaller than* ε*. The finite number of terms which may be further away from* a *than* ε *are*

$$a_1, a_2, a_3, \ldots, a_{N-1}.$$

Note that N *depends on* ε*, although we do not explicitly state this in the definition. This is because we have to choose* N *appropriately, depending on the given* ε*.*

Example 5.2.2. Let us consider the sequence $a_n = \frac{1}{n}$ for $n \in \mathbb{N}$. We would like to show that the sequence has limit 0. We will follow the definition of a limit and need to show that for each $\varepsilon > 0$ there is an N such that

$$\left|\frac{1}{n}\right| < \varepsilon \text{ for all } n \geq N.$$

We take ε as given. The condition $\frac{1}{n} < \varepsilon$ is equivalent to the condition $n > \frac{1}{\varepsilon}$. So let us try to choose N to be the next natural number larger than $\frac{1}{\varepsilon}$. Then we have that $\frac{1}{N} < \varepsilon$. With this we get the following chain of inequalities for $n \geq N$:

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

In particular, we see that $a_n = \frac{1}{n} < \varepsilon$ for all $n \geq N$. Hence we have shown that 0 is the limit of the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$.

Example 5.2.3. The sequence $1, -1, 1, -1, \ldots$ is divergent. It is interesting to prove this using the definition of limit. It requires working (implicitly or explicitly) with the negation of the defining property including the various quantifiers.

Theorem 5.2.4 (Algebra with sequences)**.** *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences. Then:*

(i) *$(a_n \pm b_n)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

(ii) *$(a_n \cdot b_n)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

(iii) *If $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$ then $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ is convergent and*

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}.$$

Exercise 5.2.5. What happens if one of the sequences (for example sequence $(a_n)_{n\in\mathbb{N}}$) is divergent? What can we say about

$$(a_n + b_n)_{n\in\mathbb{N}},$$
$$(a_n \cdot b_n)_{n\in\mathbb{N}} \quad \text{and}$$
$$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}?$$

5.3 One test for convergence

Definition 5.3.1. We will say that a sequence is *increasing [steigend] if* $a_n \geq a_m$ *whenever* $n > m$*. It is decreasing [fallend] if* $a_n \leq a_m$ *whenever* $n > m$*. A sequence is monotone [monoton] if it is either increasing or decreasing.*

Example 5.3.2.

- The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is decreasing since $\frac{1}{n} < \frac{1}{m}$ if $n > m$.
- The sequence $\left(\frac{n-1}{n}\right)_{n\in\mathbb{N}}$ is increasing.
- The sequence $((-1)^n)_{n\in\mathbb{N}}$ is neither increasing nor decreasing.

Definition 5.3.3. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ is *bounded [beschränkt] if there are some* $b_u, b_l \in \mathbb{R}$ *such that* $b_l \leq a_n \leq b_u$ *for all* n*. In this case we call* b_u *and* b_l *the upper [obere Schranke] and lower bound [untere Schranke], respectively.*

Otherwise we say that $(a_n)_{n\in\mathbb{N}}$ *is unbounded [unbeschränkt].*

Example 5.3.4.

- The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is bounded since $0 \leq \frac{1}{n} \leq 1$ for each $n \in \mathbb{N}$.
- The sequence $(2^n)_{n\in\mathbb{N}}$ is unbounded since for each $b \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $2^n > b$. *I.e.*, there is no upper bound.

Theorem 5.3.5 (Monotone Convergence Theorem)**.** *Let* $(a_n)_{n\in\mathbb{N}}$ *be a monotone sequence. If* a_n *is bounded then* a_n *converges.*

Lecture 6 — Series

6.1 Partial sums and convergence

Definition 6.1.1.

(i) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Then a *series [Reihe] is the sequence* $(s_n)_{n\in\mathbb{N}}$ *of partial sums [Partialsummen]*

$$s_n := a_1 + \ldots + a_n.$$

Usually we write $\sum_{n=1}^{\infty} a_n$ *for the sequence* $(s_n)_{n\in\mathbb{N}}$ *and call* a_n *its terms [Summanden].*

(ii) *In case the series* $(s_n)_{n\in\mathbb{N}}$ *converges to* $s \in \mathbb{R}$ *we write*

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s.$$

Remark 6.1.2. In the convergent case, the notation $\sum_{n=1}^{\infty} a_n$ has two different meanings:

- The sequence of partial sums $(a_1 + \ldots + a_n)_{n\in\mathbb{N}}$, and
- a number $s \in \mathbb{R}$, namely the limit of the partial sums; it is also called the *value [Wert] of the series.*

Example 6.1.3.

(i) *Decimal expansion [Dezimaldarstellung]: The decimal expansion of a number* $x \in \mathbb{R}$ *can be defined as a series. The partial sums are* $s_n = d_0 + \frac{d_1}{10} + \ldots + \frac{d_n}{10^n}$*, that is, finite expansions up to the* n*-th digit. The limit is* $x = \lim s_n$*. For instance* $\pi = 3.14\ldots = 3 + \frac{1}{10} + \frac{4}{100} + \ldots$*.*

(ii) *We claim* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$*, i.e., we claim for the sequence* s_n *of partial sums that*

$$s_n := \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} \quad \to \quad 1 \quad \textit{as} \quad n \to \infty.$$

Proof: Writing

$$\frac{1}{n(n+1)} = \frac{-(n^2-1)+n^2}{n(n+1)} = -\frac{n-1}{n} + \frac{n}{n+1}, \qquad \textit{for } n \in \mathbb{N},$$

we see we can apply a telescope sum trick:

$$\begin{aligned} s_n &= \Big(-0+\frac{1}{2}\Big) + \Big(-\frac{1}{2}+\frac{2}{3}\Big) + \Big(-\frac{2}{3}+\frac{3}{4}\Big) + \ldots + \Big(-\frac{n-1}{n}+\frac{n}{n+1}\Big) \\ &= -0 + \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \quad \to \quad 1 \qquad \textit{as } n \to \infty. \end{aligned}$$

Example 6.1.4. If we are careless, we can easily run into contradictions:

$$0 = (1-1)+(1-1)+\ldots = 1+(-1+1)+(-1+1)+\ldots = 1.$$

In naive language, infinite sums are not associative.

The following theorem give some necessary condition on a sequence in order for the corresponding series to converge:

Theorem 6.1.5. *If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. (We also write $a_n \to 0$ as $n \to \infty$.)*

Proof. We have $a_n = s_n - s_{n-1}$ for $n \geq 2$ and thus, using $s_n = \sum_{k=1}^{n} a_k \to s$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0. \qquad \square$$

6.2 Important examples

In Theorem 6.1.5 we have seen that the summands of a convergent series form a null sequence.

The converse, however, does not hold as the following example shows:

Example 6.2.1. The *harmonic series [harmonische Reihe]*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

has terms $a_n = \frac{1}{n}$ forming a null sequence.

Nevertheless, the sequence of partial sums is unbounded. Indeed, for $n \geq 1$ consider the subsequence [Teilfolge]

$$\begin{aligned} s_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} \\ &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{\geq 1/2} + \underbrace{\left(\frac{1}{5}+\ldots+\frac{1}{8}\right)}_{\geq 1/2} + \ldots + \underbrace{\left(\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n}\right)}_{\geq 1/2} \\ &\geq 1 + \frac{n}{2} \to \infty. \end{aligned}$$

Thus, the harmonic series does not converge.

Moreover $(s_n)_{n\in\mathbb{N}}$ is increasing, and hence our argument shows that $\sum \frac{1}{n}$ diverges to infinity; as for sequences we denote this symbolically by $\sum \frac{1}{n} = \infty$.

A very important series will turn out to be the following:

Theorem 6.2.2. *Let $x \in \mathbb{R}$. The* geometric series *[geometrische Reihe]*

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$$

converges for all $|x| < 1$ to

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

while for $|x| \geq 1$ the series diverges.

Proof. The geometric sum gives

$$s_n = \sum_{j=0}^{n} x^j = 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \qquad \text{for } x \neq 1. \tag{6.1}$$

When $|x| < 1$ we see that $x^n \to 0$ as $n \to \infty$; hence $\lim s_n = \frac{1}{1-x}$.

For $|x| \geq 1$ also $|x^n| = |x|^n \geq 1$, and so (x^n) is not a null sequence and hence $\sum x^n$ diverges by Theorem 6.1.5. □

Example 6.2.3.

- $\left|\frac{1}{2}\right| < 1$ and hence

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2.$$

- $\left|\frac{1}{3}\right| < 1$ and hence

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}.$$

- $\left|-\frac{1}{2}\right| < 1$ and hence

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \pm \ldots = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

Example 6.2.4. A periodic decimal expansion is, up to an additive constant, a geometric series; it always defines a rational number. For example,

$$\begin{aligned} 2.\overline{34} :=& 2.343434\ldots = 2 + \frac{34}{10^2} + \frac{34}{10^4} + \frac{34}{10^6} + \cdots = 2 + \frac{34}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \ldots\right) \\ =& 2 + \frac{34}{100} \cdot \frac{1}{1 - \frac{1}{100}} = 2 + \frac{34}{100} \cdot \frac{100}{99} = 2 + \frac{34}{99} = \frac{232}{99}. \end{aligned}$$

6.3 Series of real numbers

In this section we will give some criteria to test for convergence of series.

Theorem 6.3.1. *A series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ converges if and only if its partial sums are bounded.*

Proof. The assumption $a_n \geq 0$ means that the sequence of partial sums (s_n) is increasing. Therefor $s_{n+1} \geq s_n$ and so $s_n \leq s$ and hence $\sum_{n=1}^{\infty} a_n$ converges by Theorem 5.3.5. □

Example 6.3.2. Consider a decimal expansion $0.d_1d_2d_3\ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ with $d_n \in \{0,1,\ldots,9\}$. The partial sums

$$s_n = \frac{d_1}{10} + \frac{d_2}{100} + \ldots + \frac{d_n}{10^n}$$

are increasing in n and are bounded by

$$s_n \leq \frac{9}{10} + \frac{9}{100} + \ldots + \frac{9}{10^n} \stackrel{\text{geom.series}}{=} \frac{9}{10} \cdot \frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}} < \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9} = 1$$

(our estimate says that $0.99\ldots9$, with n digits, is indeed less than 1).

Thus, by Theorem 6.3.1 every decimal expansion converges.

This boundedness criterion can be used for a comparison test for convergence:

Theorem 6.3.3 (Majorisation of real series)**.** *Suppose $(x_n)_{n\in\mathbb{N}}$ is a real sequence for which there exists a convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers $a_n \geq 0$ with*

$$0 \leq x_n \leq a_n \qquad \text{for all } n \in \mathbb{N}.$$

Then $\sum_{n=1}^{\infty} x_n$ also converges and $\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} a_n$.

We say that a_n majorises [majorisiert] x_n.

Proof. Denote $C := \sum_{k=1}^{\infty} a_k$. We consider partial sums. By assumption, $\sum_{k=1}^{n} a_k \leq C$ and so

$$0 \leq \sum_{k=1}^{n} x_k \leq \sum_{k=1}^{n} a_k \leq C.$$

Thus, $\sum_{k=1}^{\infty} x_k$ converges by Theorem 6.3.1. □

Exercise 6.3.4. Suppose that for a real series $\sum_{n=1}^{\infty} a_n$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $a_n \geq x_n \geq 0$ such that $\sum_{n=1}^{\infty} x_n$ is divergent. Prove that $\sum_{n=1}^{\infty} a_n$ diverges as well.

6.4 Trigonometric and other functions

Many important functions can be defined as series:

- *The exponential function [Exponentialfunktion]:*

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- *The sine function [Sinusfunktion]:*

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- *The cosine function [Cosinusfunktion]*

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \pm \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Lecture 7 — Continuous Functions

We talked about functions in Chapter 4. Recall that a function $f : X \to Y$ assigns to every element x of the set X an element $y = f(x)$ of the set Y.

7.1 Everyday examples

Example 7.1.1.

(i) If we drive a car with constant velocity v then we know from physics that the distance s we travelled at time t equals

$$s = s_0 + v \cdot t,$$

where s_0 is the initial distance at time $t = 0$. We see immediately that s depends on the time t, *i.e.*, s is a function in t.

$$\begin{aligned} s : [0,\infty) &\longrightarrow \mathbb{R} \\ t &\longmapsto s_0 + v \cdot t \end{aligned}$$

The graph of this function is illustrated in Figure 7.1 (a).

We see that small changes of the variable t induce small changes in the distance s.

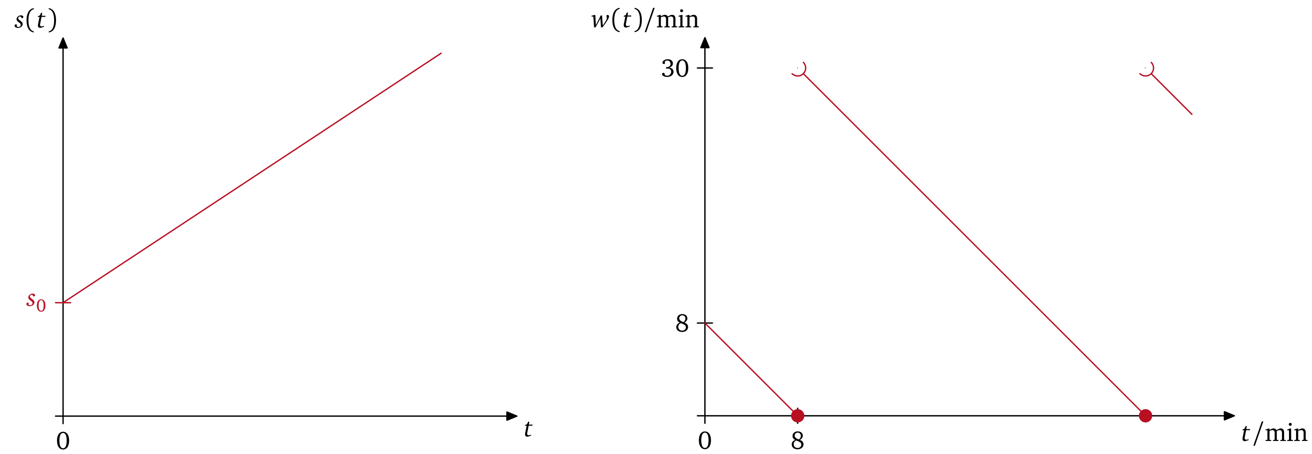
(ii) If we use the train from Darmstadt to Frankfurt and we want to catch another train in Frankfurt, then we are interested in the delay of the first train. We look at the function

$$w : [0,\infty) \longrightarrow [0,\infty)$$

which assignes to each delay t the time $w(t)$ we have to wait in Frankfurt. If we assume the first train to arrive at $xx : 48$ and the next train to depart at $xx : 56$, then a delay of 3 minutes means that we have to wait 5 minutes. And a delay of 8 minutes means that we have to wait 0 minutes. But if we have a delay of $8 + \varepsilon$ minutes, then we will not catch the train and we will have to wait for the next one (assume it departs in 30 minutes).

So a delay of 8 minutes means no waiting time. But if the delay is only a little bit more than that we will have to wait for almost 30 minutes. This means a small change in the variable t can result in a big change of the variable w.

The graph of this function is illustrated in Figure 7.1 (b).



(a) The function $s : [0, \infty) \to \mathbb{R}$ in (i) is *continuous*.

(b) The function $w : [0, \infty) \to [0, \infty)$ in (ii) is *not continuous*.

Figure 7.1: The graphs of the functions in Example 7.1.1.

7.2 Definitions of continuity

Definition 7.2.1 (ε-δ-condition)**.** A function $f : X \to Y$ is called *continuous [stetig] in a point $x_0 \in X$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\left|f(x) - f(x_0)\right| < \varepsilon \qquad \text{for all } x \in X \text{ with } \left|x - x_0\right| < \delta. \tag{7.1}$$

Equivalently, using quantifiers, this reads: $f : X \to Y$ is continuous in $x_0 \in X$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)\left|x - x_0\right| < \delta \Rightarrow \left|f(x) - f(x_0)\right| < \varepsilon.$$

Now let us have another look at our examples:

(i) *Fix some value t_0. We have*

$$\left|s(t_0) - s(t)\right| = \left|s_0 + v \cdot t_0 - (s_0 + v \cdot t)\right| = \left|v \cdot t_0 - v \cdot t\right| = \left|v(t_0 - t)\right| = |v| \cdot \left|t_0 - t\right|.$$

Now choose $\varepsilon > 0$ arbitrarily and assume that $\left|s(t_0) - s(t)\right| < \varepsilon$. We have to find $\delta > 0$ such that $\left|s(t_0) - s(t)\right| < \varepsilon$ holds for all t with $\left|t_0 - t\right| < \delta$. From the inequality

$$\left|s(t_0) - s(t)\right| = |v| \cdot \left|t_0 - t\right| < \varepsilon$$

we see that we can pick $\delta := \frac{\varepsilon}{|v|}$ and we can easily verify that this will work. This proves that s is continuous.

Note also that δ may depend on ε.

(ii) *Consider $t_0 = 8$. We will show that w is not continuous in t_0. I.e., we have to show that there is some $\varepsilon > 0$ such that for each $\delta > 0$ there is some t with $\left|t_0 - t\right| < \delta$ but $\left|w(t_0) - w(t)\right| > \varepsilon$.*

Choose $\varepsilon = 1$ and let $\delta > 0$. Set $t := t_0 + \min\left\{1, \frac{\delta}{2}\right\}$. Then $\left|t_0 - t\right| = \min\left\{1, \frac{\delta}{2}\right\} < \delta$ but $\left|w(t_0) - w(t)\right| = |w(t)| = 30 - \min\left\{1, \frac{\delta}{2}\right\} > 1$.

Hence, w is not continuous at $t_0 = 8$.

If we have a function $\mathbb{R} \to \mathbb{R}$, then there is another characterisation of continuity:

Theorem 7.2.2 (Limit test). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on a neighbourhood U of $x_0 \in \mathbb{R}$ but not necessarily defined in x_0. Then there are equivalent:*

(a) f is continuous in x_0,

(b) for every convergent sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with limit x_0 we have that

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x_0).$$

Proof. "⇒" Assume that the ε-δ-condition is satisfied in x_0. We need to show that for any sequence $x_n \to x_0$ in the domain, the image sequence satisfies $f(x_n) \to f(x_0)$. Let $\varepsilon > 0$ be arbitrary, and pick a $\delta > 0$ as in Equation (7.1) on page 38. Since $x_n \to x_0$, we can choose $N \in \mathbb{N}$ such that

$$\left|x_n - x_0\right| < \delta \qquad \text{for all } n \geq N.$$

But then Equation (7.1) implies

$$\left|f(x_n) - f(x_0)\right| < \varepsilon \qquad \text{for all } \left|x_n - x\right| < \delta,$$

which is equivalent to

$$\left|f(x_n) - f(x_0)\right| < \varepsilon \qquad \text{for all } n \geq N,$$

which implies that $f(x_n)$ converges to $f(x_0)$.

"⇐" Assume that the limit condition holds. We prove the continuity of f by contradiction.

Suppose there exists $\varepsilon > 0$ for which we cannot find $\delta > 0$ such that Equation (7.1) holds. In particular, (7.1) will not be satisfied for $\delta = \frac{1}{n}$ for any $n \in \mathbb{N}$. Thus, there exists x_n with $\left|x_n - x_0\right| < \frac{1}{n}$ such that

$$\left|f(x_n) - f(x_0)\right| > \varepsilon.$$

Hence $f(x_n)$ does not converge to $f(x_0)$. □

Example 7.2.3.

(i) Consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$. Choose $x_0 = 0$. Then the function has the limit 0 in x_0. For a proof, we need to choose any sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = 0$. Now we have to show that the sequence $f_n = x_n^2$ for $n \in \mathbb{N}$ converges to 0. This, however, is not difficult using our theorem about algebra with sequences:

$$\lim_{n\to\infty} x^2 = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = 0 \cdot 0 = 0.$$

(ii) Now let us look at a more complicated example which does not have any limit in 0. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto \sin\left(\frac{1}{x}\right)$.

Consider the sequence $x_n = \frac{1}{\pi \cdot n}$ for $n \in \mathbb{N}$. Then

$$f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin(\pi \cdot n) = 0.$$

Now consider the sequence $y_n = \frac{2}{\pi\cdot(4n+1)}$. This again gives a sequence of function values:

$$f(y_n) = \sin\left(\frac{1}{y_n}\right) = \sin\left(\frac{\pi\cdot(4n+1)}{2}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$$

So this time we get 1 as the limit of our sequence.

For two different sequences we have obtained two different limits. Therefore, by the limit test (Theorem 7.2.2), this function is not continuous in 0.

7.3 Properties of continuous functions

Theorem 7.3.1 (Algebra with continuous functions). *Let $f, g : U \to \mathbb{R}$ be two continuous functions. Then*

- $f \pm g$,
- $f \cdot g$,
- $\frac{f}{g}$, *and*
- $f \circ g$

are continuous (where they are defined).

Proof. This will be an exercise for you :-). □

Example 7.3.2. The following are examples of continuous functions:

(i) All polynomials are continuous. This follows easily from the theorem about algebra with continuous functions and from the fact that the constant functions and the identity function on $\mathbb{R}$ are continuous.

(ii) Consider the following functions:

$$\begin{aligned} \exp : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

These functions are continuous.

(iii) Rational functions are continuous on the subset of $\mathbb{R}$ where the denominator is different from 0.

The following is an important theorem about continuous functions:

7.4 The Intermediate Value Theorem

Theorem 7.4.1 (*Intermediate Value Theorem [Zwischenwertsatz]*)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous and let c be strictly between $f(a)$ and $f(b)$. Then there is an x strictly between a and b such that $f(x) = c$.*

We will not give a proof because it is very technical. See Figure 7.2 for an illustration.

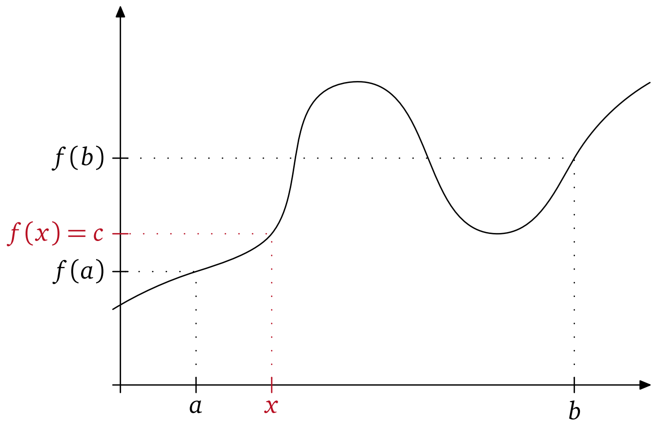


Figure 7.2: f is a continuous function $[a,b] \to \mathbb{R}$. Moreover, $f(a) < c < f(b)$ is chosen. By the intermediate value theorem there is $x \in [a,b]$ such that $f(x) = c$.

Lecture 8 — Differentiable Functions

Let $U \subseteq \mathbb{R}$ and $f : U \to \mathbb{R}$.

Assume that we want to construct the tangent [Tangente] t to the graph of f at a fixed point $x_0 \in U$. Since t goes through the point $(x_0, f(x_0))$, it suffices to determine the slope [Steigung] of t.

To this end, we first draw a line l_x through the points $(x_0, f(x_0))$ and $(x, f(x))$. This line intersects the graph of f and is not yet the required tangent. But if we now move the point x towards x_0 on the x-axis, the line l_x gets closer and closer to the tangent t. In the limit $x \to x_0$ (if it exists), the line l_x and the tangent t coincide. This process is illustrated in Figure 8.1

Now have a closer look at the slope of l_x. It is

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

Since l_x tends to t as x tends to x_0, the slope of t is the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

8.1 Definition of differentiability

Definition 8.1.1. Let $f : (a, b) \to \mathbb{R}$ be a function and $x_0 \in (a, b)$. Then f is called *differentiable [differenzierbar] in x_0 if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and coincides for all sequences $x_n \to x_0$. The derivative [Ableitung] of f in x_0 is denoted by $f'(x_0)$. (Read: f prime of x_0.)

If f is differentiable in each point of (a, b) then it is called differentiable on (a, b). In this case, f' is a function $(a, b) \to \mathbb{R}$.

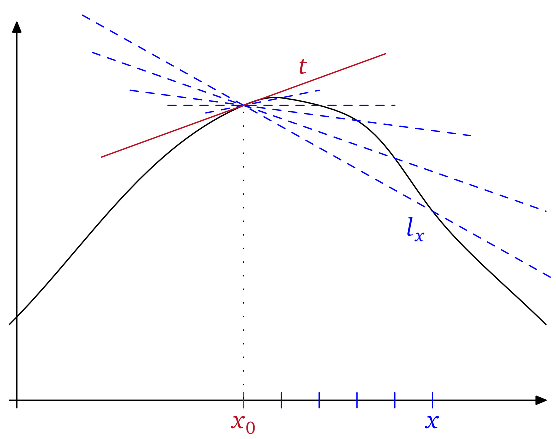


Figure 8.1: The tangent t in a point x_0 can be constructed as the limit of a sequence of secants.

(The last step is more abstract than it seems. It takes us in one stride from a single value $f'(x_0)$ to a function f')

Theorem 8.1.2. *If $f : (a,b) \to \mathbb{R}$ is differentiable in x_0 then f is also continuous in x_0.*

Proof. Let $x_0 \in (a,b)$. Then the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists since f is differentiable in x_0. As x goes to x_0, the denominator converges to zero. Hence the limit can only exist if the numerator also converges to zero. If the numerator converges to zero then $f(x)$ converges to $f(x_0)$. In other words,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

This is our definition of continuity. □

On the other hand, continuity of f does not imply differentiability as the following example will show:

Example 8.1.3. Consider the function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto |x|. \end{aligned}$$

We look at $x_0 = 0$ and show that f is continous in x_0. We choose a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ which approaches 0 from above (we denote this by $x_n \searrow 0$). Then this yields

$$\lim_{x_n \searrow 0} f(x_n) = \lim_{x_n \searrow 0} |x_n| \stackrel{x_n > 0}{=} \lim_{x_n \searrow 0} x_n = 0.$$

If we consider a sequence x_n which approaches 0 from below (we denote this by $x_n \nearrow 0$), then we get

$$\lim_{x_n \nearrow 0} f(x_n) = \lim_{x_n \nearrow 0} |x_n| \stackrel{x_n < 0}{=} \lim_{x_n \nearrow 0} -x_n = 0.$$

This shows that f is continous in x_0.

Now let us check whether f is differentiable. Again we choose a sequence $x_n \searrow 0$. This yields

$$\lim_{x_n \searrow 0} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{x_n \searrow 0} \frac{f(x_n)}{x_n} = \lim_{x_n \searrow 0} \frac{|x_n|}{x_n} \stackrel{x_n > 0}{=} \lim_{x_n \searrow 0} \frac{x_n}{x_n} = 1.$$

On the other hand, if we have a sequence $x_n \nearrow 0$, we get

$$\lim_{x_n \nearrow 0} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{x_n \nearrow 0} \frac{f(x_n)}{x_n} = \lim_{x_n \nearrow 0} \frac{|x_n|}{x_n} \stackrel{x_n < 0}{=} \lim_{x_n \nearrow 0} \frac{-x_n}{x_n} = -1.$$

We see that the limits do not coincide, which means that f is not differentiable in $x_0 = 0$.

While a differentiable function is continous, the derivative of a continous function need not be continous.

Another way to write the definition

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

of the derivative of f at x_0 is to write the sequence x_n with limit x_0 as $x_0 + h$ and look at the limit $h \to 0$. Then we get

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Now we look at a few examples and determine some derivatives:

Example 8.1.4.

(i) $f : \mathbb{R} \to \mathbb{R} : x \mapsto c \cdot x$ with $c \in \mathbb{R}$.

$$f'(x_0) = \lim_{x_n \to x_0} \frac{cx_n - cx_0}{x_n - x_0} = \lim_{x_n \to x_0} \frac{c(x_n - x_0)}{x_n - x_0} = c.$$

(ii) $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^k$.

$$f'(x_0) = \lim_{x_n \to x_0} \frac{x_n^k - x_0^k}{x_n - x_0} = \lim_{x_n \to x_0} (x_n^{k-1} + x_n^{k-2} x_0 + \ldots + x_0^{n-1}) = k x_0^{k-1}.$$

8.2 Properties of differentiable functions

Theorem 8.2.1 (Algebra with differentiable functions)**.** *Let $f, g : (a, b) \to \mathbb{R}$ be two functions differentiable in x_0. Then*

- $f \pm g$
- $f \cdot g$
- $\frac{f}{g}$
- $f \circ g$

is differentiable and the derivative is

- $(f \pm g)' = f' \pm g'$
- $(fg)' = f'g + fg'$(product rule *[Produktregel])*
- $\frac{f}{g} = \frac{f'g - fg'}{g^2}$ *(quotient rule [Quotientenregel])*
- $(f \circ g)' = f' \circ g \cdot g'$ *(chain rule [Kettenregel])*

Proof.

- $f \pm g$: Exercise.
- $f \cdot g$:

$$
\begin{aligned}
&\lim_{x_n \to x_0} \frac{(fg)(x_n) - (fg)(x_0)}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} \frac{f(x_n)g(x_n) - f(x_0)g(x_0)}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} \frac{f(x_n)g(x_n) - f(x_0)g(x_0) + \overbrace{f(x_n)g(x_0) - f(x_n)g(x_0)}^{=0}}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} \frac{f(x_n)g(x_n) - f(x_n)g(x_0) + f(x_n)g(x_0) - f(x_0)g(x_0)}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} \frac{f(x_n)(g(x_n) - g(x_0)) + (f(x_n) - f(x_0))g(x_0)}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} \frac{f(x_n)(g(x_n) - g(x_0))}{x_n - x_0} + \frac{(f(x_n) - f(x_0))g(x_0)}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} f(x_n)\frac{(g(x_n) - g(x_0))}{x_n - x_0} + g(x_0)\frac{(f(x_n) - f(x_0))}{x_n - x_0} \\
=\; &\lim_{x_n \to x_0} f(x_n) \lim_{x_n \to x_0} \frac{(g(x_n) - g(x_0))}{x_n - x_0} + \lim_{x_n \to x_0} g(x_0) \lim_{x_n \to x_0} \frac{(f(x_n) - f(x_0))}{x_n - x_0} \\
=\; &f(x_0)g'(x_0) + g(x_0)f'(x_0).
\end{aligned}
$$

- $\frac{f}{g}$: Exercise.
- $f \circ g$: *(we do the proof in the case where g is injective)*. Write

$$\frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{x_n - x_0} = \frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{g(x_n) - g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0}$$

Here we used the injectivity of g, which assures that $g(x_n) - g(x_0) \neq 0$. Now we can determine the limit:

$$
\begin{aligned}
\lim_{x_n \to x_0} \frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{x_n - x_0} &= \lim_{x_n \to x_0} \frac{(f \circ g)(x_n) - (f \circ g)(x_0)}{g(x_n) - g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(g(x_n)) - f(g(x_0))}{g(x_n) - g(x_0)} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(g(x_n)) - f(g(x_0))}{g(x_n) - g(x_0)} \lim_{x_n \to x_0} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= f'(g(x_0)) \cdot g'(x_0).
\end{aligned}
$$

□

Now we shall have a look at a useful application from everday life. At first we have the following definition:

Definition 8.2.2. Let $f : (a,b) \to \mathbb{R}$ be a function and $x_0 \in (a,b)$. Then x_0 is called

- *local minimum [lokales Minimum] if there exists an* $\varepsilon > 0$ *such that for all* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ $f(x_0) \leq f(x)$
- *local maximum [lokales Maximum] if there exists an* $\varepsilon > 0$ *such that for all* $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ $f(x_0) \geq f(x)$.
- *local extremum [lokale Extremstelle] if* x_0 *is either a local maximum or a local minimum.*

Now we can formulate the following important theorem:

Theorem 8.2.3 (Local extrema)**.** *Let* $f : (a,b) \to \mathbb{R}$ *be a differentiable function and* $x_0 \in (a,b)$. *If* f *has a local extremum in* x_0 *then* $f'(x_0) = 0$.

Proof. Let x_0 be a local extremum. Without loss of generality we may assume that x_0 is a local maximum; *i.e.*, $f(x_0) \geq f(x)$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Then

$$f'(x_0) = \lim_{h \to 0, h<0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0$$

since $f(x_0 + h) - f(x_0) \leq 0$ and $h < 0$. But on the other hand

$$f'(x_0) = \lim_{h \to 0, h>0} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0$$

since $f(x_0 + h) - f(x_0) \leq 0$ and $h > 0$. Since f is differentiable these two limits have to coincide, which yields $f'(x_0) = 0$. □

8.3 The Mean Value Theorem

Theorem 8.3.1 (*Mean Value Theorem [Mittelwertsatz]*)**.** *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function. Then there is a real number* c, $a < c < b$, *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This theorem seems rather technical, but it is beautifully illustrated by drawing a wavy graph and showing that there is a point where the tangent has the same slope as the line through the endpoints of the graph; see Figure 8.2 for an illustration.

One should point out that c *need not be unique. Also this theorem is a typical existence theorem. It tells us that something exists, but gives us no hints how to find it. Even for simple functions it might be impossible to actually determine the value of such a* c.

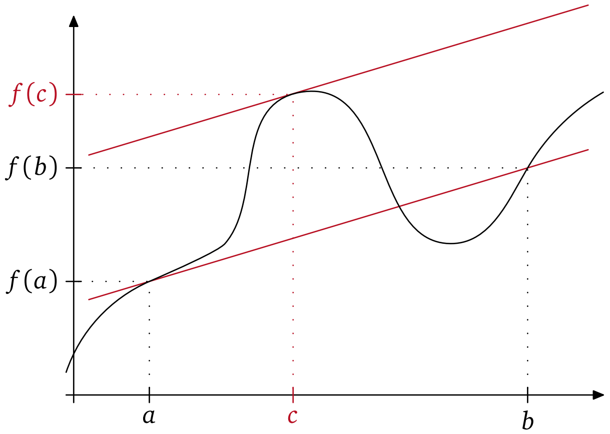


Figure 8.2: An illustration for the mean value theorem. The tangent through $(c, f(c))$ has slope $\frac{f(b)-f(a)}{b-a}$.

8.4 One application and tool: L'Hôspital's Rule

The following theorem yields another way to find limits:

Theorem 8.4.1 (*L'Hôspital's Rule [Regel von L'Hôspital]*). *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions and $x_0 \in (a, b)$. Furthermore, let $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$. We consider the function $\frac{f(x)}{g(x)}$.*

If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Remark 8.4.2.

- We can only apply this rule for limits where the variable approaches a real number, *i.e.*, *not* ∞. So if we have

$$\lim_{n\to\infty} \sin\left(\tfrac{1}{n}\right) n,$$

then we cannot apply l'Hôspital's rule. First we have to substitute the sequence by (for example) $k := \frac{1}{n}$. As n goes to infinity, k goes to 0. This yields

$$\lim_{n\to\infty} n \sin\left(\tfrac{1}{n}\right) = \lim_{k\to 0} \frac{\sin(k)}{k}.$$

Now we can apply l'Hôspital's rule and we get

$$\lim_{n\to\infty} n \sin\left(\tfrac{1}{n}\right) = \lim_{k\to 0} \frac{\sin(k)}{k} = \lim_{k\to 0} \frac{\cos(k)}{1} = 1.$$

- Also note that it is crucial that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. Consider, for instance,

$$\lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

but

$$\lim_{x \to 0} \frac{(\sin x)'}{(\cos x)'} = \lim_{x \to 0} \frac{\cos x}{-\sin x} = -\infty.$$

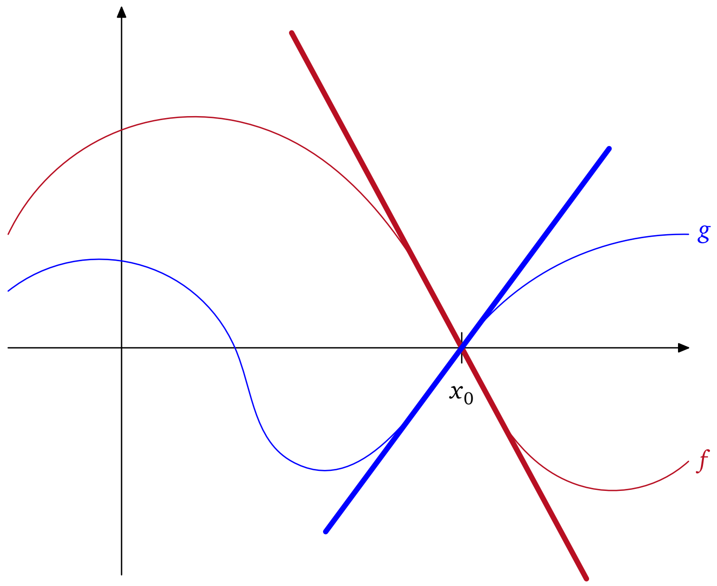


Figure 8.3: Two functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x_0) = g(x_0) = 0$. By l'Hôspital's rule, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$.

Lecture 9 — Integration

In antiquity, Archimede determined the volume of special bodies such as the cone, sphere, and cylinder. To calculate areas or volumes in general is the main task of integration. The first attempt for a systematic treatment of integration goes back to Cavalieri in the 17th century.

The integral of a function in one variable is the oriented area bounded by the graph. Two questions arise:

- *For which functions can we declare the integral?*
- *How do we compute integrals?*

The answer to the second question will be deferred until Section 9.2: The Fundamental Theorem of Calculus will turn out to be crucial.

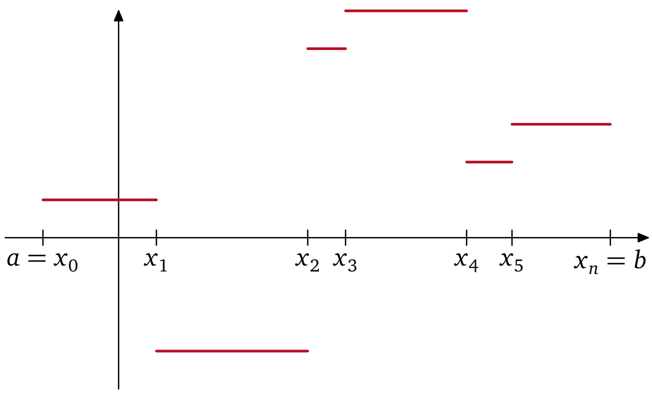
The first question has less practical impact; in fact, all functions occurring in daily life are integrable. It is, however, an interesting mathematical problem. Still, we will in this course mainly restrict our attention to continuous functions, which are always integrable.

9.1 Step functions

Before coming to continuous functions, which we are mainly interested in, we will start with another interesting and easy to handle class of functions.

Definition 9.1.1. A function $\phi : [a, b] \to \mathbb{R}$ is a *step function [Treppenfunktion] if there is a partition [Zerlegung]* $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ *of* $[a, b]$ *such that* ϕ *is constant on each interval* (x_{k-1}, x_k) *for* $k = 1, \ldots, n$*. Note that the number* n *of steps is finite and we do not constrain the values* $\phi(x_k)$*.*

An example of a step function:



Let us denote the set of step functions on $[a, b]$ *by* $S[a, b]$*. The sole point of introducing these functions is that their integrals are obvious since we know how to compute the area of a rectangle:*

Definition 9.1.2 (Integral of step functions)**.** Let $\phi \in S[a,b]$ with $\phi(x) = c_k$ on (x_{k-1}, x_k) for $k = 1, \ldots, n$. Then we set

$$\int_a^b \phi(x)\,dx := \sum_{k=1}^n c_k(x_k - x_{k-1}).$$

We also admit $a = b$, in which case the sum is empty and $\int_a^a \phi(x)\,dx = 0$.

See Figure 9.1 for an illustration.

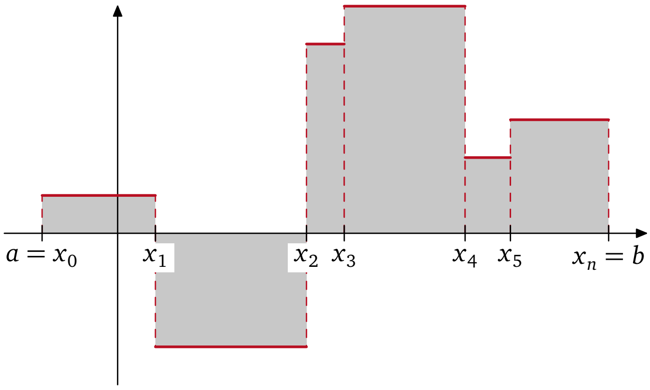


Figure 9.1: The integral over a step function.

Remark 9.1.3. The same step function can be described with respect to many different partitions, for instance, we can always include additional support points into a given partition. Then $\int_a^b \phi(x)\,dx$ remains invariant:

- For just one additional support point, this is seen as follows: If $\phi(x) := c$ on $[a,b]$ and $\xi \in (a,b)$, then

$$c(b-a) = c(\xi - a) + c(b - \xi), \tag{9.1}$$

  since rectangle areas add.

- For the general case, if X is a partition $a = x_0 < x_1 < \ldots < x_i = b$ and Y is $a = y_0 < y_1 < \ldots < y_j = b$ then their union forms a partition Z of form $a = z_0 < z_1 < \ldots < z_k = b$, having $k \leq i + j$ points. Appealing to Equation (9.1) above, we see that the sums with respect to X and Z are equal, and so are the sums with respect to Y and Z. Consequently, the sums for X and Y are also equal, which means the integral is well-defined.

9.2 The Fundamental Theorem of Calculus

Definition 9.2.1. A differentiable function $F \colon [a,b] \to \mathbb{R}$ is called a *primitive* or *antiderivative [Stammfunktion] of* $f : [a,b] \to \mathbb{R}$ *if* $F' = f$.

Example 9.2.2.

(i) For $f(x) = x^2$ the function $F(x) = \frac{1}{3}x^3$ is a primitive.

(ii) For $f(x) = e^x$ the function $F(x) = e^x$ is a primitive.

Often, the explicit form of a primitive can only be guessed. Nevertheless it always exists if f is continuous:

Theorem 9.2.3 (*Fundamental Theorem of Calculus [Fundamentalsatz der Differential- und Integralrechnung]*). *Suppose a continuous function $f:[a,b]\to\mathbb{R}$ has a primitive $F:[a,b]\to\mathbb{R}$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Let us rephrase the statement, which presents perhaps the most important fact of calculus. The equation $\left(\int_a^x f(t)\,dt\right)' = f(x)$ means that indefinite integration and differentiation are inverse operations, cancelling one another. This is not at all clear from the definition of integral and derivative!

For f constant, i.e., $f(x)=c$, this is immediate to see: If we denote $I(x) := \int_a^x f(x)\,dx$ then $I(x)=(x-a)c$ and so $I'(x)=c$.

Obviously, if F is a primitive of f, then so is $F+c$ for c constant. Conversely, any two primitives $F,G:[a,b]\to\mathbb{R}$ of the same function f satisfy

$$(F-G)' = F' - G' = f - f = 0;$$

This implies that $F-G$ is constant. That is, a primitive of f is well-defined up to a constant. Making use of this property we see that the integral of f can be computed using any of its primitives F.

The Fundamental Theorem allows us to integrate most functions introduced so far. It will be convenient to write $F(x)\big|_a^b := F(b)-F(a)$.

Example 9.2.4.

(i) From the examples for differentiation, the following is immediate:

$$\int_a^b x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_a^b$$

Thanks to the linearity of the integral (*i.e.*, $\int (f+g)\,dx = \int f\,dx + \int g\,dx$) this formula suffices to integrate polynomials.

(ii)

$$\int_a^b e^x\,dx = e^x\Big|_a^b$$

(iii)

$$\int_a^b \cos x\,dx = \sin x\Big|_a^b, \qquad \int_a^b \sin x\,dx = -\cos x\Big|_a^b.$$

(iv) Moreover,

$$\int_a^b \frac{1}{1+x^2}\,dx = \arctan x\Big|_a^b,$$

(v) and, provided $[a,b]$ does not contain a zero of cos,

$$\int_a^b \frac{1}{\cos^2 x}\,dx = \tan x\Big|_a^b.$$

9.3 Rules for integration

Each law of differentiation yields a law for integration, via the Fundamental Theorem.

Let us call a function continuously differentiable [stetig differenzierbar] if its derivative is continuous.

We consider the product rule first.

Theorem 9.3.1 (*Integration by parts [partielle Integration]*). *If $f, g\colon [a,b] \to \mathbb{R}$ are continuously differentiable then*

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx.$$

Note that the two integrals on the right hand side exist in view of our assumptions on f, g.

Proof. The function $h := fg$ can be differentiated using the product law: $h' = f'g + fg'$. In particular, h' is continuous, and so

$$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = \int_a^b h'(x)\,dx \overset{\text{Fund'l Thm.}}{=} h(x)\Big|_a^b = f(x)g(x)\Big|_a^b. \qquad \square$$

Example 9.3.2.

$$\int_a^b \cos(x)x\,dx = \sin(x)x\Big|_a^b - \int_a^b \sin(x)\cdot 1\,dx = \sin(x)x\Big|_a^b + \cos(x)\Big|_a^b$$

We now discuss the chain rule. Let us first introduce some more notation.

Suppose $F, f = F'\colon [a,b] \to \mathbb{R}$ and $x, y \in [a,b]$. Then the Fundamental Theorem gives

$$F(y) - F(x) = \int_x^y f(t)\,dt.$$

The same formula will hold for $x > y$ as well, provided we set

$$\int_x^y f(t)\,dt := -\int_y^x f(t)\,dt.$$

Theorem 9.3.3 (*Substitution*). *Let $f:[\alpha,\beta]\to\mathbb{R}$ be continuous and $\phi:[a,b]\to[\alpha,\beta]$ be continuously differentiable. Then*

$$\int_a^b f\big(\phi(t)\big)\phi'(t)\,dt=\int_{\phi(a)}^{\phi(b)} f(x)\,dx. \tag{9.2}$$

Proof. Let $F:[\alpha,\beta]\to\mathbb{R}$ be a primitive of f. According to the chain rule,

$$(F\circ\phi)'(t)=F'\big(\phi(t)\big)\phi'(t)=f\big(\phi(t)\big)\phi'(t),$$

and so (9.2) follows from

$$\int_a^b f\big(\phi(t)\big)\phi'(t)\,dt \overset{\text{Fund'l Thm.}}{=} (F\circ\phi)(t)\Big|_a^b=F\big(\phi(b)\big)-F\big(\phi(a)\big)\overset{\text{Fund'l Thm.}}{=}\int_{\phi(a)}^{\phi(b)} f(x)\,dx. \quad\square$$

Example 9.3.4.

(i) Integration is invariant under translation in the domain: For $c\in\mathbb{R}$,

$$\int_a^b f(\underbrace{t+c}_{\phi(t)})\,dt \overset{(9.2)}{=} \int_{\phi(a)=a+c}^{\phi(b)=b+c} f(x)\,dx \qquad (\phi'(t)=1).$$

(ii) For $c\in\mathbb{R}$ and $\phi(t):=ct$ we have

$$\int_a^b f(ct)c\,dt \overset{(9.2)}{=} \int_{ca}^{cb} f(x)\,dx \quad \overset{c\neq 0}{\Longrightarrow} \quad \int_a^b f(ct)\,dt=\frac{1}{c}\int_{ca}^{cb} f(x)\,dx.$$

(iii) Let us now discuss a classical problem: the area of the unit disk. The area of the upper half disk is the integral $\int_{-1}^{1}\sqrt{1-x^2}\,dx$. We want to substitute x by $\phi(t):=\sin t$ in order to take advantage of the identity $\sin^2 t+\cos^2 t=1$. Note that $\phi:[-\frac{\pi}{2},\frac{\pi}{2}]\to[-1,1]$ is continuously differentiable and invertible. Substitution gives

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx=\int_{\phi^{-1}(-1)}^{\phi^{-1}(1)}\underbrace{\sqrt{1-\sin^2 t}}_{\sqrt{\cos^2 t}}\,\underbrace{(\sin t)'}_{\cos t}\,dt=\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt \overset{Ex.}{=} \frac{\pi}{2}.$$

Here, we used the fact that $\cos t\geq 0$ for $t\in[-\frac{\pi}{2},\frac{\pi}{2}]$. Thus the unit disk has area π.

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