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Introductory Course Mathematics

Exercise Sheet 8 with hints

G34 Determine the tangent at x_0 :

(a) $f(x) = 2x^3 - 7, \quad x_0 = -1$

(b) $f(x) = \frac{1}{x}, \quad x_0 = \frac{1}{2}$

SOLUTION: In each case we will describe the tangent as a function $t : \mathbb{R} \to \mathbb{R} : x \mapsto t(x)$. We use the equation

$$\frac{t(x) - t(x_0)}{x - x_0} = f'(x_0)$$

to determine the equation for the tangent:

$$t(x) = f'(x_0)(x - x_0) + f(x_0),$$

since $t(x_0) = f(x_0)$.

(a) $f'(x_0) = 6x_0 = -6, f(x_0) = -9$ and hence

$$t(x) = -6x - 6 - 9 = -6x - 15.$$

(b) $f'(x_0) = -\frac{1}{x_0^2} = -4, f(x_0) = 2$ and hence

$$t(x) = -4x + 2 + 2 = 4x + 4.$$

G35 Does $\lim\limits_{x \to x_0} \dfrac{f(x_0) - f(x)}{x_0 - x}$ exist for the following function?

$$f(x) = \left|x^3\right|, \quad x_0 = 0$$

Use the definition of differentiability to decide whether the function is differentiable in $x_0 = 0$.

SOLUTION:

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x} &= \lim_{x \to x_0} \frac{\left|x_0^3\right| - |x|^3}{x_0 - x} \\
&= \lim_{x \to x_0} \frac{|x|^3}{x} \\
&= \begin{cases} \lim\limits_{x \to x_0} \frac{x^3}{x} = \lim\limits_{x \to x_0} x^2 = 0 & \text{if } x \searrow 0 \\ \lim\limits_{x \to x_0} \frac{-x^3}{x} = \lim\limits_{x \to x_0} -x^2 = 0 & \text{if } x \nearrow 0 \end{cases}
\end{aligned}$$

Thus, the limit exists and hence f is differentiable in $x_0 = 0$.

G36 Prove from the definition of differentiability:

(a) If $f(x) = x^2$, then $f'(x) = 2x$.

(b) If $f(x) = x^3$, then $f'(x) = 3x^2$.

(c) If $f(x) = x^n$, for $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.

(d) If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$.

SOLUTION: Use the definition via the limit.

G37 Write the following function as a composition of simpler functions and calculate their derivatives using the chain rule: $f(x) = \sqrt{(2x^2 + x)^3 + 1}$

SOLUTION: Let $g : \mathbb{R} \to \mathbb{R} : x \mapsto x^3 + 1$ and $h : \mathbb{R} \to \mathbb{R} : x \mapsto 2x^2 + x$. Then

$$f = \sqrt{\cdot} \circ g \circ h$$

and

$$f'(x) = \frac{3(2x^2 + x)^2(4x + 1)}{2\sqrt{(2x^2 + x)^3 + 1}}.$$

G38 Prove using the defintion by power series from Lectures 6 and 7:

(a) If $f(x) = e^x$ then $f'(x) = e^x$.

(b) If $f(x) = \sin x$ then $f'(x) = \cos x$.

(c) If $f(x) = \cos x$ then $f'(x) = -\sin x$.

SOLUTION: (a) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 0 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$ and hence

$$\begin{aligned} f'(x) &= (0)' + \sum_{n=1}^{\infty} \left(\frac{x^n}{n!} \right)' = 0 + \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= e^x \end{aligned}$$

(b) $f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and hence

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)' = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x \end{aligned}$$

(c) $f(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 0 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ and hence

$$\begin{aligned} f'(x) &= (0)' + \sum_{n=1}^{\infty} \left((-1)^n \frac{x^{2n}}{(2n)!} \right)' = 0 + \sum_{n=1}^{\infty} (-1)^n \frac{2nx^{2n-1}}{(2n)!} \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \\ &= -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = -\sin x \end{aligned}$$

G39 Compute the derivatives of the following functions:

(a) $f_1(x) = x^4 - x^2 + 5x - 7$

(b) $f_2(x) = \frac{x^2+5}{\sqrt{x^2-7x+1}}$

(c) $f_3(x) = x^2 e^{x^2}$

(d) $f_4(x) = 2^x$

(e) $f_5(x) = x^x$

SOLUTION: (a) $f_1'(x) = 4x^3 - 2x + 5$

(b) $f_2'(x) = \dfrac{2x \cdot \sqrt{x^2 - 7x + 1} - (x^2 + 5)\frac{2x-7}{2\sqrt{x^2-7x+1}}}{x^2 - 7x + 1}$

(c) $f_3'(x) = 2xe^{x^2} + x^2 e^{x^2} 2x$

(d) $f_4(x) = 2^x = e^{x \ln 2}$ and hence $f_4'(x) = e^{x \ln 2} \ln 2 = 2^x \cdot \ln 2$

(e) $f_5(x) = x^x = e^{x \ln x}$ and hence $f_5'(x) = e^{x \ln x}(\ln x + x \cdot \frac{1}{x}) = x^x(\ln x + 1)$

G40 Show, that $(f \pm g)' = f' \pm g'$.

SOLUTION:

$$\begin{aligned}(f \pm g)'(x_0) &= \lim_{x \to x_0} \frac{(f \pm g)(x) - (f \pm g)(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x) \pm g(x) - f(x_0) \mp g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(f(x) - f(x_0)) \pm (g(x) - g(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) \pm g'(x_0)\end{aligned}$$

G41 Use the product rule and the chain rule to prove the quotient rule.

SOLUTION: $\frac{f}{g} = f \cdot \frac{1}{g}$ and $\frac{1}{g} = h \circ g$, where $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto \frac{1}{x}$.
Thus we get

$$rcl \left(\frac{f}{g}\right)' \overset{\text{product rule}}{=} f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' \overset{\text{chain rule}}{=} f' \cdot \frac{1}{g} + f \cdot \frac{-g'}{g^2} = \frac{f'g - fg'}{g^2}$$

G42 Decompose a fixed real number c into two summands such that their product is maximal.

SOLUTION: Let $c = x + y$. Then $y = c - x$ and we want to maximise the function $f : \mathbb{R} \to \mathbb{R} : f(x) = x(c - x)$. We compute $f'(x) = -2x + c$. By the theorem on local extrema, the local maximum $x_{\max}$ must satisfy $f'(x_{\max}) = 0$ and hence $x_{\max} = \frac{c}{2}$.

Now check that this is indeed a maximum by comparing the function value with the function values of points in the neighbourhood.