Fachbereich Mathematik
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Introductory Course Mathematics

Exercise Sheet 6 with hints

G23 (Mini-Test) Decide whether the following statements are true or false:

(a) If $(a_n)_{n\in\mathbb{N}}$ is a null sequence then $\sum_{n=0}^{\infty} a_n$ converges.

(b) For $|x| < 1$ the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

(c) The series $1 - 1 + 1 - 1 + 1 - 1 \pm \dots$ converges to 0.

SOLUTION: (a) This is false, consider the harmonic series $a_n = \frac{1}{n}$.

(b) This is true.

(c) This is false.

G24 (Convergence I) Compute the values of the following series (if they converge):

(a) $\sum\limits_{n=0}^{\infty} \left(\frac{99}{100}\right)^n$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} \pm \dots$

(c) $5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \dots$

(d) $\sum\limits_{k=0}^{\infty} \left(\frac{1}{2^k} + \left(-\frac{1}{3}\right)^k\right)$

SOLUTION: Use the formula for geometric series.

G25 (Convergence II) Which of the following series converge? Prove your answer!

(a) $\sum\limits_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right)$

(b) $\sum\limits_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

(c) $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$

(d) $\sum\limits_{k=1}^{\infty} \frac{1}{k!}$

SOLUTION: (a) This is a telescope sum. The partial sums are $s_n = \sum\limits_{k=1}^{n} \left(\sqrt{k} - \sqrt{k-1}\right) = -\sqrt{0} + \sqrt{n} = \sqrt{n}$ and hence the series diverges.

(b) Again, this is a telescope sum. The partial sums are $\sum\limits_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{1} = \frac{1}{n+1} = 1 - \frac{1}{n+1}$ and hence the series converges to 1.

(c) $\sum_{n=1}^{\infty} \frac{1}{2n} = 2 \sum_{n=1}^{\infty} \frac{1}{n}$ and hence the series diverges (since the harmonic series diverges).

(d) Since $\left(\frac{1}{2}\right)^k \geq \frac{1}{k!}$ for all $k \geq 2$ the series $\sum_{k=1}^{\infty} \frac{1}{k!}$ is majorised by $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$, which converges.

G26 Recall Theorem 5.2.4 from Friday:

Theorem 5.2.4 Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences. Then:

(a) $(a_n \pm b_n)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

(b) $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

(c) If $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$ then $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

What happens if one of the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ is divergent? What can we say about

$$(a_n + b_n)_{n\in\mathbb{N}},$$
$$(a_n \cdot b_n)_{n\in\mathbb{N}} \quad \text{and}$$
$$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}?$$

SOLUTION:

- $(a_n + b_n)_{n\in\mathbb{N}}$ is divergent. (Use the negation of the definition of convergence to prove this.)
- $(a_n \cdot b_n)_{n\in\mathbb{N}}$: Assume without loss of generality that a_n convergences and b_n diverges. There are several cases to distinguish:
  - If $\lim_{n\to\infty} a_n = a \neq 0$ then $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is divergent.
  - If $\lim_{n\to\infty} a_n = 0$ and b_n is bounded, then $\lim_{n\to\infty} a_n \cdot b_n = 0$.
  - Otherwise anything can happen. *E.g.* if $a_n = \frac{1}{n}$ and $b_n = n$ then $a_n b_n$ converges (to 1). But if $a_n = \frac{1}{n}$ and $b_n = n^2$ then $a_n b_n$ diverges.
- $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$: Write $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$ and do the same case distinction as above (but being careful with the denominator).

G27 Let $\sum_{n=0}^{\infty} a_n$ with $a_n \geq 0$ for every n be a convergent series. Prove that every reordering of this series converges.

(Note that the assumption $a_n \geq 0$ is crucial here; without this assumption the statement is false.)

SOLUTION: Prove (for example by contradiction) that every reordering is bounded. Then the claim follows from the monotone convergence theorem.

G28 (A puzzle) Assume that n students attended the lecture this afternoon. Furthermore, assume that, when arriving, some of them shook hands with some others. Note that not everyone necessarily shook hands with everyone else and there might even be people who did not shake hands with anyone.

Prove: No matter who actually shook hands with whom, there are always two students who shook the same number of hands.

Solution: Any given student shook between 0 and $n-1$ hands. (There are n possibilities hence.) Assume any student shook a different number of hands. But there cannot at the same time be a student who shook 0 hands and one who shook $n-1$ hands. Thus, there are two students who shook the same number of hands.