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Introductory Course Mathematics

Exercise Sheet 5 with hints

G19 (Limits I)

(a) Consider the sequence

$$a_n = \frac{2n-3}{5n+7}, \quad n \in \mathbb{N}.$$

(i) Show that the limit of this sequence is $\frac{2}{5}$.

(ii) Which terms of the sequence are closer to $\frac{2}{5}$ than $\varepsilon = \frac{1}{10}$?

(b) (i) What is the limit of the sequence $a_n = \frac{1}{2^n}$ for $n \in \mathbb{N}$?

(ii) What is the limit of the sequence

$$\frac{1}{2}, \quad \frac{1}{2}+\frac{1}{4}, \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}, \ldots$$

Can you give a geometric interpretation of this limit process?

(c) The first terms of an infinite sequence are $1, 3, 7, 15, 31, 63$.

(i) Find a recursive definition for the sequence.

(ii) Find an explicit definition.

(d) Find a recursive definition for the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, \quad \ldots$$

What is the limit of this sequence?

Solution:

(a) (i)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2n-3}{5n+7}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{2-\frac{3}{n}}{5+\frac{7}{n}} = \frac{2}{3}$$

(ii) Solve the inequality $\left|\frac{2n-3}{5n+7}-\frac{2}{5}\right| \leq \frac{1}{10}$ for n.

(b) (i) Shown (for example by induction) that $2^n > n$. This implies $0 < \frac{1}{2^n} < \frac{1}{n}$. Thus, the limit is 0.

(ii) Set

$$a_n := \sum_{k=1}^{n} \frac{1}{2^k}.$$

Then

$$a_n = 1 - \frac{1}{2^n}.$$

Therefore, the limit is 1.

Consider a round cake. Adding up the sum above means that we start out with half the cake. Then we add half of the rest to that ($\frac{1}{4}$), then again half of the remainder ($\frac{1}{8}$) and so on. This illustrates that the limit is 1.

(c) (i) $a_1 = 1$, $a_{n+1} = 2a_n + 1$

(ii) $a_n = 2^n - 1$

(d) $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$.

First show that the sequence is convergent. (It is increasing and bounded.) Then show that the limit a satisfies $a = \sqrt{2a}$, which has solutions 0 and 2. But 0 cannot be the limit since $(a_n)_{n\in\mathbb{N}}$ is increasing and $0 < a_1$. Thus, $\lim_{n\to\infty} a_n = 2$.

G20 (Limits II)

Determine the limit (if it exists) of

$$a_n = \frac{5}{n} + \frac{7n}{n^2+1}, \qquad b_n = \left(6 + \frac{1}{n}\right)\left(\frac{n+2}{2n+1} - 1\right), \qquad c_n = \frac{2n^2-2}{3n^2-3},$$

$$d_n = \frac{\frac{1}{n^2} + \frac{1}{n^3}}{\frac{1}{n} + \frac{1}{n^2}}, \qquad e_n = \frac{2n + (-1)^n n}{n+1}.$$

Solution: Apply standard arguments.

G21 (Limits III)

Determine the limit (if it exists) of

(a) $a_n = \sqrt{n^2+1} - n, \quad n \in \mathbb{N}$.

(b) $b_n = n(\sqrt{n^2+1} - n), \quad n \in \mathbb{N}$.

(c) $c_n = n^2(\sqrt{n^2+1} - n), \quad n \in \mathbb{N}$.

Solution:

(a)

$$\begin{aligned} a_n = \sqrt{n^2+1} - n &= \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n} \\ &= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} = \frac{\frac{1}{n}}{\sqrt{n + \frac{1}{n^2}} + 1} \to 0 \end{aligned}$$

(b)

$$b_n = \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{(1+\frac{1}{n^2})}+1} \to \frac{1}{2}$$

(c)

$$c_n = \frac{n^2}{\sqrt{(n^2+1)}+n} = \frac{n}{\sqrt{(1+\frac{1}{n^2})}+1} > \frac{n}{\sqrt{1}+1} \to \infty$$

G22 (The Fibonacci Sequence)

Consider the closed form for the Fibonacci sequence as given in the lecture:

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

(a) Prove that f_n is a natural number for $n = 1, 2, 3$.

(b) Prove that it is a natural number for every $n \in \mathbb{N}$.

SOLUTION:

(a) Simple calculations show that $f_1 = f_2 = 1$ and $f_3 = 2$.

(b) Prove the recursion formula for the Fibonacci numbers, *i.e.*, $f_{n+1} = f_n + f_{n-1}$. Then the claim follows by induction.