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Introductory Course Mathematics

Exercise Sheet 4 with hints

G14 (Injectivity, Surjectivity, Bijectivity I)

(a) Which of the following functions are injective, surjective, bijective?

$$\begin{aligned} f_1 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned} \qquad \begin{aligned} f_4 : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$

$$\begin{aligned} f_2 : \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned} \qquad \begin{aligned} f_5 : \mathbb{R} \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto \frac{1}{x} \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto x^2 \end{aligned} \qquad \begin{aligned} f_6 : \mathbb{R} \setminus \{0\} &\to \mathbb{R}_{>0} \\ x &\mapsto \frac{1}{x^2} \end{aligned}$$

Also determine the image of each function.

(b) Find a function $f : \mathbb{N} \to \mathbb{N}$ which is

(i) injective but not surjective,

(ii) surjective but not injective.

SOLUTION:

(a)

	f_1	f_2	f_3	f_4	f_5	f_6
injective	–	×	×	×	×	–
surjective	–	–	×	×	–	×
bijective	–	–	×	×	–	–

$$\begin{aligned} \operatorname{im} f_1 &= f_1(\mathbb{R}) = \mathbb{R}_{\geq 0} \\ \operatorname{im} f_2 &= f_2(\mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0} \\ \operatorname{im} f_3 &= f_3(\mathbb{R}_{\geq 0}) = \mathbb{R}_{\geq 0} \end{aligned} \qquad \begin{aligned} \operatorname{im} f_4 &= f_4(\mathbb{R}) = \mathbb{R} \\ \operatorname{im} f_5 &= f_5(\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \{0\} \\ \operatorname{im} f_6 &= f_6(\mathbb{R} \setminus \{0\}) = \mathbb{R}_{>0} \end{aligned}$$

(b) (i) $f : \mathbb{N} \to \mathbb{N}, x \mapsto x + 1$

(ii) $f : \mathbb{N} \to \mathbb{N}, 1 \mapsto 1, x \mapsto x - 1$ for $x > 1$.

G15 (Composition of Functions)

(a) Find functions f and g such that the following functions can be written as $f \circ g$.

$$F_1(x) = \sqrt{x+9} \qquad F_3(x) = \sqrt{x} + 2$$
$$F_2(x) = (x-5)^2 \qquad F_4(x) = \tfrac{1}{x-1}$$

(b) Consider the functions f and g from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = x^2$ and $g(x) = x - 3$. Find the composite functions $f \circ f$, $f \circ g$, $g \circ f$ and $g \circ g$ and determine the domain of each function. Demonstrate that $f \circ g$ is not necessarily the same as $g \circ f$.

(c) Find $f \circ g \circ h$ where $f(x) = x/(x+1)$, $g(x) = x^2$ and $h(x) = x + 3$. Find the maximal subset of $\mathbb{R}$ on which $f \circ g \circ h$ is defined.

SOLUTION:

(a) (i) $f(x) = \sqrt{x}$, $g(x) = x + 9$.

(ii) $f(x) = x^2$, $g(x) = x - 5$.

(iii) $f(x) = x + 2$, $g(x) = \sqrt{x}$.

(iv) $f(x) = \frac{1}{x}$, $g(x) = x - 1$.

(b) $(f \circ f)(x) = x^4$. The domain of $f \circ f$ is $\mathbb{R}$.

$(g \circ f)(x) = x^2 - 3$. The domain of $g \circ f$ is $\mathbb{R}$.

$(f \circ g)(x) = (x - 3)^2$. The domain of $f \circ g$ is $\mathbb{R}$.

$(g \circ g)(x) = x - 6$. The domain of $g \circ g$ is $\mathbb{R}$.

(c)

$$(f \circ g \circ h)(x) = \frac{(x+3)^2}{(x+3)^2 + 1}.$$

G16 (Preimages)

Determine the set

$$\{x \in \mathbb{R} \mid f(x) = 1\},$$

with

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 - x^2 - 4x + 5 \end{aligned}$$

SOLUTION: Determine the solutions of $x^3 - x^2 - 4x + 5 = 1$.

$$\{x \in \mathbb{R} \ : \ f(x) = 1\} = \{1, 2, -2\}.$$

G17 (Zeroes)

Find all zeroes of the following functions:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 - 6x^2 + 11x - 6 \\ g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^4 - 4x^3 + 6x^2 - 4x + 1 \\ h : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^4 - 1 \end{aligned}$$

G18 (Bonus Exercise: Injectivity, Surjectivity, Bijectivity II) Let $f : X \to X$ be a function. Prove that f is

(a) injective,

(b) surjective,

(c) bijective

if and only if $f \circ f$ is.

SOLUTION: We did this in the lecture.