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Introductory Course Mathematics

Exercise Sheet 3 with hints

G7 (Proof by Contradiction)

Consider (again) the proposition $(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B \Rightarrow f)$ and explain why this justifies proofs by contradiction.

SOLUTION: The equivalence of the propositions $(A \Rightarrow B)$ and $(A \wedge \neg B \Rightarrow \mathbf{f})$ means that instead of proving the first proposition, we can prove the second proposition.

For proving the second proposition we assume that A is true and B is false. From this we need to derive a wrong conclusion. This is exactly the technique of the proof by contradiction.

G8 (Direct Proof)

Show by a direct proof that for all $a, b \in \mathbb{R}$ the equation $a + \frac{1}{a} = b$ implies $a^3 + \frac{1}{a^3} = b^3 - 3b$.

SOLUTION: We compute

$$b^3 = (a + \frac{1}{a})^3 = a^3 + \frac{1}{a^3} + 3(a + \frac{1}{a}) = a^3 + \frac{1}{a^3} + 3b.$$

The statement follows.

G9 (Contraposition)

Show by proving the contraposition: For all $x \in \mathbb{R}$ we have that $x > 0$ implies the inequality $\frac{3x-4}{2x+4} > -1$.

SOLUTION: We show that $\frac{3x-4}{2x+4} \leq -1$ implies $x \leq 0$:

$$\frac{3x-4}{2x+4} \leq -1 \quad \Rightarrow \quad 3x - 4 \leq -2x - 4 \quad \Rightarrow \quad 5x \leq 0 \quad \Rightarrow \quad x \leq 0.$$

(Note that the first implication is not an equivalence since x might be -2 in the second inequality but not in the first.)

G10 (Irrationality of $\sqrt{2}$)

Analyse the proof that $\sqrt{2}$ is not a rational number. Why is this a proof by contradiction and not a proof by contraposition?

SOLUTION: The proof assumes that there is a rational number q such that $q^2 = 2$. The fact that q is a rational number is also used in the proof. At the end of the proof we do not conclude that q is not rational but we reach the false conclusion that the numerator and the denominator of q have a common factor.

Therefore it is a proof by contradiction and not the proof of the contraposition.

G11 (Induction)

Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

SOLUTION: Induction start: For $n = 1$ we get

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1(1+2)(2 \cdot 1 + 1)}{6}.$$

Induction step: Assume we know that the claim holds for some value of n, *i.e.*,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6} \\
&= \frac{(n+1)\left(2n^2 + 7n + 6\right)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.
\end{aligned}$$

G12

(a) Let n be a natural number. Show that n^2 is even if and only if n is even.

(b) Show that $x^2 = 6$ does not have a rational solution.

(c) Show that $1 + \sqrt{2}$ is not a rational number. Show that $a + b\sqrt{2}$ is not rational for rational numbers a and b with $b \neq 0$.

(d) Show that $x^3 = 2$ does not have a rational solution.

SOLUTION:

(a) If n is even, it is divisible by 2. Then n^2 is also divisible by 2.

If n is odd, it is of the form $2k - 1$ for a natural number k. Then $n^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$ is obviously odd.

(b) We follow the proof that $x^2 = 2$ does not have a rational solution. Set $x = \frac{a}{b}$ with a and b having no common factor. The key observation is that the equation $a^2 = 6b^2$ implies that a is even. Then we get $a = 4c$ for some c and $2c^2 = 3b^2$. This equation implies that b is even which is a contradiction.

(c) If $a = 0$, then we have to consider $r = b\sqrt{2}$. If r is rational, then $\sqrt{2} = \frac{r}{b}$ is also rational.

Assume $a \neq 0$. Then $r = (a + b\sqrt{2})^2 = a^2 + 2ab\sqrt{2} + 2b^2$ and $\sqrt{2} = \frac{r - a^2 - 2b^2}{2ab}$. If r is rational, then $\sqrt{2}$ is also rational.

(d) Use the proof that shows that $x^2 = 2$ does not have a rational solution.

G13 (Bonus Exercise: What is wrong?)

Assume the following equation for a complex number x:

$$x^2 + x + 1 = 0.$$

Then

$$x^2 = -1 - x.$$

If we assume that $x \neq 0$, we can divide by x, which yields

$$x = -\frac{1}{x} - 1.$$

Substituting this expression for x in the original equation leads to

$$\begin{aligned} x^2 + \left(-\frac{1}{x} - 1\right) + 1 &= 0 \\ x^2 - \frac{1}{x} &= 0 \\ x^2 &= \frac{1}{x} \\ x^3 &= 1 \\ x &= 1. \end{aligned}$$

Substituting $x = 1$ in the original equation yields

$$3 = 0.$$

SOLUTION: The equation $z^3 = 1$ has three solutions over $\mathbb{C}$. The other two solutions are $-\frac{1}{2} \pm \sqrt{\frac{3}{4}}$, which are also solutions for $x^2 + x + 1 = 0$.

The wrong solution $z = 1$ is introduced by substituting $x = -\frac{1}{x} - 1$ into the original equation. (This step increases the degree of the equation by 1 and hence adds a new solution.)