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WS 2010/11 28.9.2010

Introductory Course Mathematics

Exercise Sheet 2 with hints

G4 (Complex Numbers)

(a) Verify that the inversion formula

$$(a+bi)^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

is correct.

(b) Compute $\frac{5-3i}{3+2i}$.

(c) Try to find the solution of $(2-i)\cdot(2-2i)$ geometrically.

(d) Let $c = a+bi$ be a complex number. Compute $(x-c)(x-\bar{c})$. Can you guess a solution for $x^2-2x+2=0$?

SOLUTION:

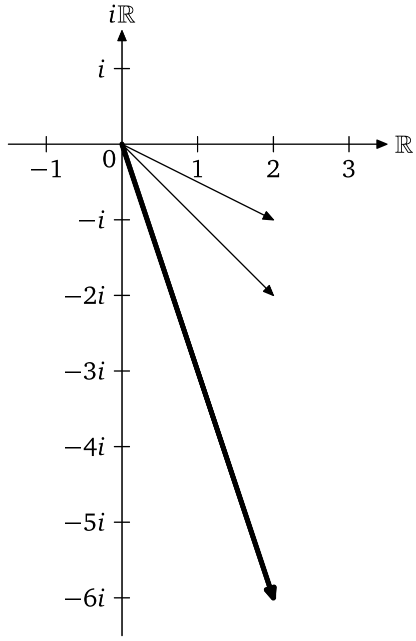
(a)

$$\begin{aligned}(a+bi)(a+bi)^{-1} &= (a+bi)\cdot\left(\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i\right)\\ &= a\frac{a}{a^2+b^2} + a\frac{-b}{a^2+b^2}i + bi\frac{a}{a^2+b^2} + b\frac{-b}{a^2+b^2}i^2\\ &= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1\end{aligned}$$

(b)

$$\frac{5-3i}{3+2i} = (5-3i)(3+2i)^{-1} = (5-3i)\frac{3-2i}{3^2+2^2} = \frac{9-19i}{13}.$$

(c) We see that $2-2i$ encloses an angle of 45° with the real line. So multiplying with $2-2i$ is a rotation about 45° clockwise and a dilation of $2-2i = \sqrt{8} \approx 2.8$. So we get:



(d)
$$(x-c)(x-\bar{c}) = x^2 - (c+\bar{c})x + c\bar{c} = x^2 - 2ax + (a^2+b^2).$$
If $a = b = 1$ this turns into $x^2 - 2x + 2$ which hence has the solutions $1+i$ and $1-i$.

G5 (Truth Tables)

Let A and B be propositions. Show that the following statments are true by setting up truth tables in each case:

(a) A is the same as $\neg(\neg A)$.

(b) $\neg(A \wedge B)$ is the same as $\neg A \vee \neg B$.

(c) $\neg(A \vee B)$ is the same as $\neg A \wedge \neg B$.

(d) $A \Rightarrow B$ is the same as $\neg A \vee B$.

Solution: The truth tables are:

(a)

A	$\neg A$	$\neg(\neg A)$
t	f	t
f	t	f

(b)

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$
t	t	t	f	f	f	f
t	f	f	t	f	t	t
f	t	f	t	t	f	t
f	f	f	t	t	t	t

(c)

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$
t	t	t	f	f	f	f
t	f	t	f	f	t	f
f	t	t	f	t	f	f
f	f	f	t	t	t	t

(d)

A	B	$A \Rightarrow B$	B	$\neg A$	$\neg A \vee B$
t	t	t	t	f	t
t	f	f	f	f	f
f	t	t	t	t	t
f	f	t	f	t	t

G6 (Propositions and Quantifiers)

(a) Show: For propositions A and B we have that $A \wedge (A \Rightarrow B)$ implies B. Interpret this rule.

(b) Find examples of implications that are not equivalences and explain why the conclusion works in one direction only.

(c) Prove the proposition

$$(A \implies B) \iff (\neg B \implies \neg A)$$

(d) Negate the following propositions:

(i) All mathematicians are smokers.

(ii) No student likes to go to parties.

(iii) All bananas are yellow.

(iv) There exists a black swan.

(e) Negate the following propositions:

(i) $\exists x \in S : A(x) \wedge B(x)$

(ii) $\forall x \in S : A(x) \Rightarrow B(x)$

(f) Write the following propositions as a formal expression using quantifiers, negate the formal expression and convert the negation back into everyday language.

(i) For every real number x there exists a natural number n such that $n > x$.

(ii) There is no rational number x satisfying the equation $x^2 = 0$.

SOLUTION:

(a) Set up a truth table.

The implication $A \Rightarrow B$ says that B is true provided A is also true. It does not say anything about the truth of A and therefore, we do not know if B is true or not.

If we know in addition that A is true, then it follows that B is also true.

(b) If x is a positive real number, then x^2 is also positive.
If the integer n is divisible by 6, then n is divisible by 3.
If it is midnight in Darmstadt, the sun does not shine.

(c) The truth table looks as follows:

A	B	$A \Rightarrow B$	$\neg B$	$\neg A$	$\neg B \implies \neg A$
t	t	t	f	f	t
t	f	f	t	f	f
f	t	t	f	t	t
f	f	t	t	t	t

(d) (i) There is a mathematician who is not a smoker.

(ii) There is a student who likes to go to parties.

(iii) There is a banana that is not yellow.

(iv) All swans are not black.

(e) (i) $\forall x \in S : \neg A(x) \vee \neg B(x)$

(ii) $\exists x \in S : A(x) \wedge \neg B(x)$

(f) (i) $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N} : n > x$

Negation: $\exists x \in \mathbb{R}\ \forall n \in \mathbb{N} : n \leq x$

There is an x in $\mathbb{R}$ such that x is larger than or equal to n for all $n \in \mathbb{N}$.

(ii) $\neg(\exists q \in \mathbb{Q} : q^2 = 0)$, which is the same as $\forall q \in \mathbb{Q} : q^2 \neq 0$.

Negation: $\exists q \in \mathbb{Q} : q^2 = 0$

There is a rational number x satisfying the equation $x^2 = 0$.