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Introductory Course Mathematics

Exercise Sheet 10 with hints

G48 Show that the union of two countable sets is countable.

SOLUTION: List the elements of the first set: $x_1, x_2, x_3, \ldots$. List the elements of the second set: $y_1, y_2, y_3, y_4, \ldots$. Now we list the elements of the union: $x_1, y_1, x_2, y_2, x_3, y_3, \ldots$.

G49 Show that any subset of a countable set is countable.

SOLUTION: If $x_1, x_2, x_3, x_4, x_5, \ldots$ is a listing of the elements of a set, then we simply delete the elements which are not in the subset and get a listing of the subset.

G50 Show that the set $\mathbb{N} \times \mathbb{N} = \{(m, n) \mid m, n \in \mathbb{N}\}$ of all pairs of natural numbers is countable. Deduce that the set of pairs $\{(x, y) \mid x \in X, y \in Y\}$ is countable if X and Y are both countable sets.

SOLUTION:

This is done by Cantor's first diagonal process. This is essentially the same proof as for the countability of rational numbers.

If $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, y_4, \ldots$ are listings of the elements of X and Y, respectively, then Cantor's first diagonal process applied to the indices of the elements yields the proof.

G51 Show that the set of all infinite sequences consisting of 0s and 1s is uncountable.

SOLUTION: This is done by Cantor's second diagonal process. If we have a list of all sequences $s_1, s_2, s_3, \ldots$, then we construct a new sequence (x_n) as follows: We set $x_i = 1$ if the i-th element of s_i is 0 and we set $x_i = 0$ if the i-th element of s_i is 1. The (x_n) is not in the list because its i-th term is different from the i-th term of sequence s_i. Therefore, the listing is incomplete.

G52 Reexamine Cantor's first diagonal process for the countability of the rational numbers. How often does it list each rational number?

SOLUTION: Each rational number is listed infinitely often, for example 2 occurs as $\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \ldots$.

G53 A rational number can also be written as an infinite decimal expansion. Try to apply Cantor's second diagonal process to the set of rational numbers and try to show that the set of rational numbers is uncountable (which is false, of course). Where does the argument break down?

SOLUTION: The new number which is constructed is indeed not on the list of rational numbers but the new number is not a rational number. Note that the decimal expansion of a rational number is periodic from a certain decimal digit on, for example

$$\frac{431}{3500} = 0.123\;142857\;142857\;142857\ldots$$

The constructed number missing from the list of rational numbers will not have that property.