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Introductory Course Mathematics

Exercise Sheet 1 with hints

NOTE: These exercises are to be solved in small teams of three to four students each. Discuss your solutions among each other. If you have questions, first see if one of your fellow students can answer your question, then ask your tutor. Teamwork is an important part of mathematics!

Rearrange the tables and chairs in the seminar room so that you can sit around the table, face each other and discuss. It is a good idea to put two or three tables together to form one large table.

G1 (Natural Numbers and Divisibility)

(a) Prove that 7 is not a divisor of 100.

(b) List all the divisors of 12, 140 and 1001. Prove that your list of divisors for 12 is complete.

(c) Use the Sieve of Eratosthenes to find all primes less than 50. What is the earliest point at which you have obtained all primes less than 50?

(d) Prove that each natural number n is divisible by 1 and n.

(e) Let p, q and r be natural numbers.

Prove: If p is a divisor of q and q is a divisor of r, then p is a divisor of r.

(f) Let n be a natural number.

Prove: If $d|n$ with $d \geq \sqrt{n}$, then there is a divisor e of n with $e \leq \sqrt{n}$.

How can you use this to simplify the test for primality by trial division?

Compare this result with your observation about the Sieve of Eratosthenes.

(g) Show: If d is a divisor of m and of n, then d is a divisor of $m + n$, $m - n$ and d^2 is a divisor of $m \cdot n$.

SOLUTION: (a) We show this by inspecting the multiples of 7 and observing that 100 is not among them. Therefore, 7 is not a divisor of 100:

$$7, 14, 21, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, \ldots$$

(b) The set of divisors of 12 is $\{1, 2, 3, 4, 6, 12\}$.

Any divisor of 12 is smaller than or equal to 12. We need to show that 5, 7, 8, 9, 10 and 11 are not divisors of 12.

We show the assertion for 5: The multiples of 5 are $5, 10, 15, \ldots$ and we see that 12 is not among them. Therefore, 5 is not a divisor of 12. It is easy to treat the other non-divisors of 12 in the same way.

The set of divisors of 140 is $\{1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, 140\}$.

The set of divisors of 1001 is $\{1, 7, 11, 13, 77, 91, 143, 1001\}$.

(c) The following table is the sieve after the multiples of 7 have been crossed out.

	2	3	$\cancel{4}$	5	$\cancel{6}$	7	$\cancel{8}$	$\cancel{9}$	$\cancel{10}$
11	$\cancel{12}$	13	$\cancel{14}$	$\cancel{15}$	$\cancel{16}$	17	$\cancel{18}$	19	$\cancel{20}$
$\cancel{21}$	$\cancel{22}$	23	$\cancel{24}$	$\cancel{25}$	$\cancel{26}$	$\cancel{27}$	$\cancel{28}$	29	$\cancel{30}$
31	$\cancel{32}$	$\cancel{33}$	$\cancel{34}$	$\cancel{35}$	$\cancel{36}$	37	$\cancel{38}$	$\cancel{39}$	$\cancel{40}$
41	$\cancel{42}$	43	$\cancel{44}$	$\cancel{45}$	$\cancel{46}$	47	$\cancel{48}$	$\cancel{49}$	$\cancel{50}$

We see that all remaining numbers are primes and can stop the process.

Observe that when you cross out the multiples of a number n, then the first multiple of n that is not crossed out yet is n^2. Why is that the case?

(d) We write $n = 1 \cdot n$. We compare this with the definition of divisibility and see that there is a number d (which is n in this case) such that $n = d \cdot 1$. We also see that there is another number d (which is 1 in this case) such that $n = d \cdot n$.

(e) If p is a divisor of q, then there is a natural number d_1 such that $q = d_1 \cdot p$. Likewise, there is a natural number d_2 such that $r = d_2 \cdot q$. Substituting the first equation into the second gives: $r = d_2 d_1 \cdot p$ and we see that p divides r.

(f) Let $d \geq \sqrt{n}$. As d divides n, there is an m such that $dm = n$. Then

$$d \cdot m = n = \sqrt{n} \cdot \sqrt{n} \leq d \cdot \sqrt{n}.$$

Cancelling d on both ends gives $m \leq \sqrt{n}$.

If we have checked that a number n is not divisible by any of the natural numbers (different from 1) smaller than or equal to $\sqrt{n}$, then it cannot be divisible by any of the numbers strictly between 1 and n. This means that we can stop the method of trial division at $\sqrt{n}$.

This coincides with the observation that Eratosthenes' Sieve in Problem 3 could be stopped when the multiples of the prime 7 were crossed out. The next prime, 11, is larger than $\sqrt{50}$.

(g) If $m = dq$ and $n = dr$, then $m + n = dq + dr = d(q + r)$. Hence d divides $m + n$. Write $mn = dq \cdot dr = d^2 \cdot qr$.

G2 (Sums and Binomial Coefficients)

(a) Calculate the follwing sums:

(i) $\sum_{i=0}^{3} i$ (ii) $\sum_{j=1}^{4} 3^j$ (iii) $\sum_{k=1}^{2} 2 + k$ (iv) $\sum_{k=1}^{2} 2 + j$

(b) Rewrite using the sigma sign:

(i) $1 + 3 + 5 + 7 + 9$

(ii) $2 + 4 + 6 + 8 + 10$

(iii) $2 + 4 + 8 + 16$

(c) Show:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(d) Show:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \text{and} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: Use the Binomial Theorem.

SOLUTION: (a)

$$\begin{aligned}
\sum_{i=0}^{3} i &= 0+1+2+3=6 \\
\sum_{j=1}^{3} 3^j &= 3+9+27=39 \\
\sum_{k=1}^{2} 2+k &= 3+4=7 \\
\sum_{k=1}^{2} 2+j &= (2+j)+(2+j)=4+2j
\end{aligned}$$

(b)

$$\begin{aligned}
1+3+5+7+9 &= \sum_{i=0}^{4} 2i+1 \\
2+4+6+8+10 &= \sum_{i=1}^{5} 2i \\
2+4+8+16 &= \sum_{i=1}^{4} 2^i
\end{aligned}$$

(c) Direct computation:

$$\begin{aligned}
\binom{n}{k}+\binom{n-1}{k} &= \frac{n!}{(n-(k-1))!(k-1)!}+\frac{n!}{(n-k)!k!} \\
&= \frac{n!}{(n-k+1)!(k-1)!}+\frac{n!}{(n-k)!k!} \\
&= \frac{n!\cdot k}{(n-k+1)!k!}+\frac{n!(n-k+1)}{(n-k+1)!k!} \\
&= \frac{n!\cdot k+n!(n-k+1)}{(n-k+1))!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k}
\end{aligned}$$

(d) From the binomial formula: Substitute $x=1$ and $y=1$ into $(x+y)^n$.

From the binomial formula: Substitute $x=1$ and $y=-1$ into $(x+y)^n$.

G3 (Real Numbers)

Let a, b and c be real numbers and ε a positive real number.

(a) Show that $|a| \le c$ is the same as saying $-c \le a \le c$.

(b) Show that $a \le |a|$ and $-|a| \le a$.

(c) Prove the triangle inequality: $|a+b| \le |a|+|b|$. *Hint:* Use the previous two inequalities.

(d) Prove the inequality $|a|-|b| \le |a-b|$.

(e) Show that $|x-a| \le \varepsilon$ is the same as saying $a-\varepsilon \le x \le a+\varepsilon$. Interpret this geometrically! What is the set of all x satisfying this condition?

(f) Determine the solutions of the inequalities $|4 - 3x| > 2x + 10$ and $|2x - 10| \leq x$.

SOLUTION: (a) We distinguish two cases:

(i) $a \geq 0$. Then $|a| = a$ and $-a \leq 0 \leq a$. Thus

$$|a| \leq c \Leftrightarrow a \leq c \Leftrightarrow a \leq c \text{ and } -c \leq -a \leq a \Leftrightarrow -c \leq a \leq c.$$

(ii) $a < 0$. Then $|a| = -a$ and $a < 0 < -a$. Thus

$$|a| \leq c \Leftrightarrow -a \leq c \Leftrightarrow -a \leq c \text{ and } -c \leq a < -a \Leftrightarrow -c \leq a \leq c.$$

(b) Use (a) with $c = |a|$ and note that $|a| \leq |a|$ is always true.

(c) By (b) we have $a \leq |a|$ and $b \leq |b|$ thus $a + b \leq |a| + |b|$. Likewise we have $-|a| \leq a$ and $-|b| \leq b$ thus $-|a| - |b| \leq a + b$. Combining this yields $-(|a| + |b|) \leq a + b \leq |a| + |b|$ which by (a) is equivalent to $|a + b| \leq |a| + |b|$

(d) Set $c := a - b$. By (c) we have $|a| = |b + c| \leq |b| + |c| = |b| + |a - b|$.

(e) By (a) $|x - a| \leq \varepsilon$ is the same as saying $-\varepsilon \leq x - a \leq \varepsilon$. Adding a to this inequality yields the desired statement.

Geometrically, when looking at the number line, this indicates all real numbers (points on the number line) that are at distance at most ε from a. So, the x satisfying it form a line segment, the closed interval $[a - \varepsilon, a + \varepsilon]$.

(f) To find the solutions of $\varepsilon 4 - 3x > 2x + 10$ we distinguish between two cases:

(i) $4 - 3x \geq 0$: Equivalentely, $x \leq \frac{4}{3}$. Then the above inequality becomes

$$4 - 3x > 2x + 10 \Leftrightarrow -6 > 5x \Leftrightarrow x < -\frac{6}{5} \leq \frac{4}{3}.$$

(ii) $4 - 3x < 0$: Equivalentely, $x > \frac{4}{3}$. Then the above inequality becomes

$$-(4 - 3x) > 2x + 10 \Leftrightarrow x > 14 > \frac{4}{3}.$$

Hence the solutions are those real number which are either greater than 14 or less than $-\frac{6}{5}$.

The second problem can be solved in a similar fashion, showing that $\frac{10}{3} \leq x \leq 10$ is the set of solutions there.