Fachbereich Mathematik
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Introductory Course Mathematics

Exercise Sheet 6

G23 (Mini-Test) Decide whether the following statements are true or false:

(a) If $(a_n)_{n\in\mathbb{N}}$ is a null sequence then $\sum_{n=0}^{\infty} a_n$ converges.

(b) For $|x| < 1$ the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$.

(c) The series $1 - 1 + 1 - 1 + 1 - 1 \pm \ldots$ converges to 0.

G24 (Convergence I) Compute the values of the following series (if they converge):

(a) $\sum\limits_{n=0}^{\infty} \left(\frac{99}{100}\right)^n$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} \pm \ldots$

(c) $5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \ldots$

(d) $\sum\limits_{k=0}^{\infty} \left(\frac{1}{2^k} + \left(-\frac{1}{3}\right)^k\right)$

G25 (Convergence II) Which of the following series converge? Prove your answer!

(a) $\sum\limits_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right)$

(b) $\sum\limits_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

(c) $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$

(d) $\sum\limits_{k=1}^{\infty} \frac{1}{k!}$

G26 Recall Theorem 5.2.4 from Friday:

Theorem 5.2.4 Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent sequences. Then:

(a) $(a_n \pm b_n)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

(b) $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

(c) If $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$ then $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}.$$

What happens if one of the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ is divergent? What can we say about

$$(a_n + b_n)_{n\in\mathbb{N}},$$
$$(a_n \cdot b_n)_{n\in\mathbb{N}} \quad \text{and}$$
$$\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}?$$

G27 Let $\sum_{n=0}^{\infty} a_n$ with $a_n \geq 0$ for every n be a convergent series. Prove that every reordering of this series converges.

(Note that the assumption $a_n \geq 0$ is crucial here; without this assumption the statement is false.)

G28 (A puzzle) Assume that n students attended the lecture this afternoon. Furthermore, assume that, when arriving, some of them shook hands with some others. Note that not everyone necessarily shook hands with everyone else and there might even be people who did not shake hands with anyone.

Prove: No matter who actually shook hands with whom, there are always two students who shook the same number of hands.