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Introductory Course Mathematics

Exercise Sheet 3

G7 (Proof by Contradiction)

Consider (again) the proposition $(A \Rightarrow B) \Leftrightarrow (A \wedge \neg B \Rightarrow f)$ and explain why this justifies proofs by contradiction.

G8 (Direct Proof)

Show by a direct proof that for all $a, b \in \mathbb{R}$ the equation $a + \frac{1}{a} = b$ implies $a^3 + \frac{1}{a^3} = b^3 - 3b$.

G9 (Contraposition)

Show by proving the contraposition: For all $x \in \mathbb{R}$ we have that $x > 0$ implies the inequality $\frac{3x-4}{2x+4} > -1$.

G10 (Irrationality of $\sqrt{2}$)

Analyse the proof that $\sqrt{2}$ is not a rational number. Why is this a proof by contradiction and not a proof by contraposition?

G11 (Induction)

Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

G12

(a) Let n be a natural number. Show that n^2 is even if and only if n is even.

(b) Show that $x^2 = 6$ does not have a rational solution.

(c) Show that $1 + \sqrt{2}$ is not a rational number. Show that $a + b\sqrt{2}$ is not rational for rational numbers a and b with $b \neq 0$.

(d) Show that $x^3 = 2$ does not have a rational solution.

G13 (Bonus Exercise: What is wrong?)

Assume the following equation for a complex number x:

$$x^2 + x + 1 = 0.$$

Then

$$x^2 = -1 - x.$$

If we assume that $x \neq 0$, we can divide by x, which yields

$$x = -\frac{1}{x} - 1.$$

Substituting this expression for x in the original equation leads to

$$\begin{aligned} x^2 + \left(-\frac{1}{x} - 1\right) + 1 &= 0 \\ x^2 - \frac{1}{x} &= 0 \\ x^2 &= \frac{1}{x} \\ x^3 &= 1 \\ x &= 1. \end{aligned}$$

Substituting $x = 1$ in the original equation yields

$$3 = 0.$$