STOCHASTIC NAVIER-STOKES EQUATIONS

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Contents

The following notes provide a brief introduction into the theory of stochastic partial differential equations with particular emphasis on its applications to stochastic equations arising in mathematical fluid dynamics, in particular stochastic Navier-Stokes equations.

The reader of these notes should be familiar with the theory of abstract evolution equations together with its applications to nonlinear partial differential equations. On the contrary we only assume a moderate knowledge in probability theory. Most of the necessary background from stochastic analysis is provided throughout the exposition. Consequently, the notes are divided up into the following three sections:

Section 1 contains a brief introduction into the theory of stochastic integration w.r.t. (cylindrical) Wiener processes on Hilbert spaces. The exposition follows closely Chapter 2 of the monograph [18].

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Section 2 introduces stochastic evolution equations on Hilbert spaces. The existing theory for these equations can essentially be divided up into three parts:

(a) the theory of martingale solutions (see [2]),
(b) the theory of mild solutions based on the semigroup approach (see [3]),
(c) the theory of variational solutions (see [19]).

For those readers that are familiar with the theory of abstract evolution equations, the semigroup approach to stochastic partial differential equations is closest and thus the natural starting point. We therefore start Section 2 with a short summary of the theory of mild solutions to semilinear stochastic evolution equations. The special case of stochastic partial-differential equations with additive noise is considered in more detail. In particular, this part of the theory is applied to the 2D-stochastic Navier-Stokes equations. For the 3D-case, however, the semigroup approach is not sufficient, so that we continue Section 2 with an introduction to the martingale approach.

Section 3 finally deals with the application of the theory of martingale solutions to equations from mathematical fluid dynamics, in particular the 3D-stochastic Navier-Stokes equations. This part follows closely the paper [9] by Flandoli, resp. some parts of the introductory course [8] by Flandoli. We also shortly discuss the existence of stationary martingale solutions and invariant measures.

These notes, of course, can only give a first rough introduction to stochastic partial differential equations and its applications to stochastic equations in mathematical fluid dynamics. In particular, we do not touch the recent progress on uniqueness of invariant measures by Kuksin and Shirikyan ([17], Hairer and Mattingly ([12]), work by Flandoli and Romito on Markov selections of martingale solutions of the 3D-stochastic Navier-Stokes equations ([11]) and very promising looking progress on the stabilizing effets of noise on the transport equation ([10]).

For more detailed surveys on the subject we refer the reader to the introductory course [8] by Flandoli and to the lecture notes [16] by Kuksin for the particular 2D-case.

1. Stochastic Integration on Hilbert spaces

1.1. **Gaussian measures on Hilbert spaces.** Let $(U, \langle\, ,\, \rangle_U)$ and $(H, \langle\, ,\, \rangle_H)$ be two separable real Hilbert spaces. Basic examples we have in mind are

(i) $L^2(\Omega, \mathcal{A}, \mu) =: L^2(\mu)$, $\langle f, g\rangle = \int_\Omega fg\, d\mu$. Note that if Ω is a separable metric space and $\mathcal{A} = \mathcal{B}(\Omega)$ the Borel-σ-algebra on Ω then $L^2(\mu)$ is separable.

(ii) $\ell^2 := \{(u_k)_{k\geq 1} \subset \mathbb{R} \mid \sum_{k=1}^\infty u_k^2 < \infty\}$, $\langle u, v\rangle := \sum_{k=1}^\infty u_k v_k$

Definition 1.1. A probability measure μ on $(U, \mathcal{B}(U))$ is called Gaussian if for all $v \in U$ the linear mapping

$$\ell_v : U \to \mathbb{R}\,, u \mapsto \langle u, v\rangle$$

has a Gaussian distribution, i.e., there exists $m(v) \in \mathbb{R}$, $\sigma(v) \in \mathbb{R}_+$ with

$$\int_U e^{it\ell_v}\, d\mu = e^{itm(v)-\frac{1}{2}t^2\sigma^2(v)}\,, t \in \mathbb{R}\,.$$

Remark 1.2. (i) $\sigma(v) > 0$ implies

$$\mu_{\ell_v}(A) := \mu(\ell_v \in A) = \frac{1}{\sqrt{2\pi\sigma(v)^2}} \int_A e^{-\frac{(x-m(v))^2}{2\sigma(v)^2}}\, dx\,, A \in \mathcal{B}(\mathbb{R})\,.$$

(ii) $\sigma(v) = 0$ implies $\mu_{\ell_v} = \delta_{m(v)}$ (= Dirac measure in $m(v)$).

Theorem 1.3. *A probability measure μ on $(U, \mathcal{B}(U))$ is Gaussian if and only if*

$$\int_U e^{i\langle u,v\rangle_U}\, \mu(du) = e^{i\langle m,v\rangle_U - \frac{1}{2}\langle Qv,v\rangle_U}\,, \forall\, v \in U\,,$$

where

- *$m \in U$ is the mean,*
- *$Q \in L(U)$ symmetric, positive semidefinite, finite trace, i.e.,*

$$\mathrm{tr}\,(Q) := \sum_{k=1}^\infty \langle Qe_k, e_k\rangle_U < \infty$$

for one (hence any) complete orthonormal system (= CONS) $(e_k)_{k\geq 1}$ of U, is the covariance operator.

In the following, $N(m, Q)$ will denote the Gaussian measure with mean m and covariance operator Q. The reason for calling m the mean and Q the covariance is provided by the following formulas (i) and (ii):

(i) $\int_U \langle x, h\rangle_U\, \mu(dx) = \langle m, h\rangle_U$
(ii) $\int_U (\langle x, h\rangle_U - \langle m, h\rangle_U)(\langle x, g\rangle_U - \langle m, g\rangle_U)\, \mu(dx) = \langle Qh, g\rangle_U$
(iii) $\int_U \|x - m\|_U^2\, \mu(dx) = \mathrm{tr}\,(Q)\,.$

Example 1.4. (1) **Wiener measure**

Let $(\beta(t))_{t\geq 0}$ be a one-dimensional Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $(\beta(t))_{t\geq 0}$ is a family of random variables such that:

(i) $\beta(0) = 0$ $\mathbb{P}$-a.s.,
(ii) for $0 \leq t_1 < t_2 < \ldots < t_n$, the increments

$$\beta(t_{i+1}) - \beta(t_i)\,, 1 \leq i \leq n-1\,,$$

are independent, $N(0, t_{i+1} - t_i)$-distributed,

(iii) $t \mapsto \beta(t)$ is continuous $\mathbb{P}$-a.s.

Restricting to a finite time interval $[0, T]$, we obtain a measurable mapping

$$\beta : \Omega \to C([0,T];\mathbb{R}) \subset L^2([0,T])\,.$$

The distribution

$$\mu(A) := \mathbb{P}(\beta \in A)\,, A \in \mathcal{B}(L^2([0,T]))$$

is called the (one-dimensional) Wiener measure. μ is a centered Gaussian measure (i.e. the mean is equal to zero) with covariance operator

$$\begin{aligned}\langle Qg, h\rangle_{L^2([0,T])} &= \int_0^T \int_0^T g(s)h(t) s \wedge t\, ds\, dt \\ &= \langle (-\Delta)^{-1} g, h\rangle_{L^2([0,T])}\end{aligned}$$

where Δ denotes the Laplace operator on $L^2([0,T])$ with Dirichlet boundary condition in 0 and Neumann boundary condition in T.

Proof:

(i)

$$\begin{aligned}\int_{L^2([0,T])} \langle x, h\rangle_{L^2([0,T])}\, \mu(dx) &= \mathbb{E}\left(\int_0^T h(s)\beta(s)\, ds\right) \\ &= \int_0^T h(s)\mathbb{E}(\beta(s))\, ds = 0\,.\end{aligned}$$

(ii)

$$\begin{aligned}\int \langle x, g\rangle \langle x, h\rangle\, \mu(dx) &= \mathbb{E}\left(\int_0^T \beta(s)g(s)\, ds \int_0^T \beta(t)h(t)\, dt\right) \\ &= \int_0^T \int_0^T g(s)h(t)\mathbb{E}(\beta(s)\beta(t))\, ds\, dt \\ &= \int_0^T \int_0^T g(s)h(t) s\wedge t\, ds\, dt\,.\end{aligned}$$

(2) **Brownian bridge measure**

The process

$$\beta^0(t) := \beta(t) - \frac{t}{T}\beta(T)\,, \quad 0 \le t \le T\,,$$

satisfies $\beta^0(T) = 0$, hence describes a Brownian bridge from 0 to 0. The distribution

$$\mu^0(A) := \mathbb{P}(\beta^0(0:T) \in A)\,, A \in \mathcal{B}(L^2([0,T]))$$

is sometimes also called the pinned Wiener measure. μ^0 is a centred Gaussian measure with covariance operator

$$\begin{aligned}\langle Q^0 g, h\rangle_{L^2([0,T])} &= \int_0^T \int_0^T g(s)h(t)(s\wedge t - \frac{st}{T})\,ds\,dt \\ &= \langle (-\Delta_D)^{-1} g, h\rangle_{L^2([0,T])}\end{aligned}$$

where Δ_D denotes the Dirichlet Laplacian on $L^2([0,T])$.

Proof:

$$\begin{aligned}&\int_{L^2([0,T])} \langle x, g\rangle\langle x, h\rangle\,\mu(dx) \\ &\qquad = \int_0^T \int_0^T g(s)h(t)\mathbb{E}(\beta^0(s)\beta^0(t))\,ds\,dt\end{aligned}$$

and

$$\mathbb{E}(\beta^0(s)\beta^0(t)) = s\wedge t - \frac{st}{T}\,.$$

Remark 1.5. On a finite-dimensional Hilbert space U any linear operator has a finite trace, hence for any symmetric and positive semidefinite Q there exists a Gaussian measure $N(0,Q)$, in particular for $Q = I$. Note that the measure $N(0,I)$ is invariant under rotations.

Such a rotationally invariant Gaussian measure μ cannot exist on an infinite dimensional Hilbert space U because of the folllowing reason: suppose on the contrary that such a measure μ would exist and let $(e_k)_{k\geq 1}$ be an ONS in U. Then the set of balls $B_{\frac{1}{4}}(e_k)$ of radius $\frac{1}{4}$ with center e_k, $k\geq 1$, is a sequence of pairwise disjoint subsets of U, all contained in the larger ball $B_2(0)$. Because of rotational invariance of μ it follows that $\mu(B_{\frac{1}{4}}(e_k)) = \mu(B_{\frac{1}{4}}(e_1))$ for $k = 1,2,3,\ldots$, and thus

$$\infty > \mu(B_2(0)) \geq \mu\left(\bigcup_{k\geq 1} B_{\frac{1}{4}}(e_k)\right) = \sum_{k=1}^{\infty} \mu(B_{\frac{1}{4}}(e_k)) = \infty\,,$$

which is a contradiction. So necessarily, $\sum_{k=1}^{\infty} \mu(B_{\frac{1}{4}}(e_k)) < \infty$ for any (Gaussian) measure on U. Since the volume of $B_{\frac{1}{4}}(e_k)$ is essentially linked to $\langle Qe_k, e_k\rangle_U$, the trace condition on Q drops out as a natural condition.

Theorem 1.6. *Let $m\in U$, $Q\in L(U)$ be symmetric, positive semidefinite with finite trace. Let $(e_k)_{k\geq 1}$ be a CONS of U consisting of eigenvectors of Q with eigenvalues $(\lambda_k)_{k\geq 1}$. Then the following statements are equivalent:*

(i) *A U-valued random variable X on $(\Omega,\mathcal{F},\mathbb{P})$ is Gaussian with mean m and covariance operator Q.*

(ii) *X can be represented as the (infinite) series*

$$X = \sum_{k=1}^{\infty} \sqrt{\lambda_k} Y_k e_k + m \,,$$

where $(Y_k)_{k\geq 1}$ are independent, $N(0,1)$-distributed random variables.

1.2. **Wiener processes on Hilbert spaces.** Q as in Section 1.1, in particular $\operatorname{tr}(Q) < \infty$.

Definition 1.7. A U-valued stochastic process $(W(t))_{t\in[0,T]}$, on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a (standard) Q-Wiener process if:

(i) $W(0) = 0$ $\mathbb{P}$-a.s.
(ii) for $0 \leq t_1 < t_2 < \ldots < t_n$, the increments

$$W(t_{i+1}) - W(t_i)\,, 1 \leq i \leq n-1\,,$$

are independent, $N(0, (t_{i+1} - t_i)Q)$-distributed
(iii) $t \mapsto W(t)$ is continuous $\mathbb{P}$-a.s.

Theorem 1.8. *(Canonical representation of a Q-Wiener process)*
Let $(e_k)_{k\geq 1}$ be a CONS of U consisting of eigenvectors of Q with eigenvalues $(\lambda_k)_{k\geq 1}$. Then $(W(t))_{t\geq 0}$ is a Q-Wiener process if and only if

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k\,, t \in [0,T]\,, \tag{1}$$

for independent Brownian motions $(\beta_k(t))_{t\geq 0}$, $k = 1, 2, \ldots$. The infinite series (1) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; C([0,T]; U))$, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} \left\| \sum_{k=1}^{n} \sqrt{\lambda_k} \beta_k(t) e_k - W(t) \right\|_U^2 \right) = 0\,.$$

An increasing family of sub-σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a filtration. $\mathcal{F}_t$ is interpreted as the information available at time t.

Definition 1.9. A Q-Wiener process $(W(t))_{t\in[0,T]}$ is called a Q-Wiener process w.r.t. a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, if

(i) $(W(t))_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted,
(ii) the increment $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t \leq T$.

Any Q-Wiener process $(W(t))_{t\in[0,T]}$ is a Q-Wiener process w.r.t. the following filtration

$$\mathcal{F}_t := \bigcap_{s>t} \tilde{\mathcal{F}}_s^0$$

where

$$\tilde{\mathcal{F}}_t^0 := \sigma(\mathcal{F}_t^0 \cup \mathcal{N})$$

and $\mathcal{N} = \{A \in \mathcal{F} \mid \mathbb{P}(A) = 0\}$ denotes the set of $\mathbb{P}$-null sets and $\mathcal{F}_t^0 = \sigma\{W(s) \mid s \in [0,t]\}$ the σ-algebra generated by the Wiener process $(W(t))_{t\in[0,T]}$.

The σ-algebra $(\mathcal{F}_t)_{t\geq 0}$, given as above, is right-continuous, i.e.,

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \qquad \forall t \in [0,T[$$

and complete, i.e., $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Such a filtration is called a natural filtration.

1.3. **Martingales on Banach spaces.** Let $(E, \|\cdot\|)$ be a (real) separable Banach space, μ be a finite measure on $(\Omega, \mathcal{F})$. Recall the definition of the E-valued Bochner integral $\int f\, d\mu$, and let

$$\mathcal{L}^1(\mu; E) := \{f : \Omega \to E \mid f \text{ (strongly) measurable}, \int_E \|f\|\, d\mu < \infty\},$$

$L^1(\mu; E)$ be the space of all μ-equivalence classes of $\mathcal{L}^1(\mu; E)$. Similarly, $\mathcal{L}^p(\mu; E)$ and $L^p(\mu; E)$.

Remark 1.10. (i) $\|\int f\, d\mu\| \leq \int \|f\|\, d\mu$ (Bochner's inequality)
(ii) $L \in L(E,F) \Rightarrow L(\int f\, d\mu) = \int L \circ f\, d\mu$ (Linearity)
(iii) main theorem of calculus:

$$f \in C([a,b]; E) \quad \Rightarrow \quad f(t) - f(s) = \int_s^t f'(r)\, dr\,, a \leq s < t \leq b\,.$$

Conditional expectations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X \in \mathcal{L}^1(\mathbb{P}; E)$, $\mathcal{F}_0 \subset \mathcal{F}$ be a sub-σ-algebra. Then

$$\exists \quad X_0 := \mathbb{E}(X \mid \mathcal{F}_0) : \Omega \to E$$

$\mathcal{F}_0$-measurable, with

$$\int_{A_0} X_0\, d\mathbb{P} = \int_{A_0} X\, d\mathbb{P} \qquad \forall\, A_0 \in \mathcal{F}_0\,,$$

and $\|\mathbb{E}(X \mid \mathcal{F}_0)\| \leq \mathbb{E}(\|X\| \mid \mathcal{F}_0)$.

In the following we fix a filtration $(\mathcal{F}_t)_{t\geq 0}$ on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.11. An E-valued stochastic process $(M_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called an $(\mathcal{F}_t)_{t\geq 0}$-martingale, if

(i) $\mathbb{E}(\|M_t\|) < \infty \qquad \forall t \geq 0$
(ii) M_t is $\mathcal{F}_t$-measurable $\qquad \forall t \geq 0$
(iii) $\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s \qquad \forall 0 \leq s \leq t$.

Remark 1.12. $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale if and only if

$$\forall \ell \in E' : \quad (\ell(M_t))_{t\geq 0} \quad \text{is a real-valued } (\mathcal{F}_t)_{t\geq 0} - \text{martingale} .$$

Theorem 1.13. *(Doob's maximal inequality)*
Let $(M_t)_{t\geq 0}$ be a right-continuous martingale, then

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]} \|M_t\|^p\right)\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\mathbb{E}\left(\|M_T\|^p\right)^{\frac{1}{p}} \qquad \forall p > 1 .$$

It follows that for $p > 1$

$$\mathcal{M}_T^p := \{(M_t)_{t\in[0,T]} \mid (M_t)\, E - \text{valued}, (\mathcal{F}_t) - \text{martingale, continuous},$$
$$\|M\|_{\mathcal{M}_T^p} := \sup_{t\in[0,T]} \mathbb{E}(\|M_t\|^p)^{\frac{1}{p}} = \mathbb{E}(\|M_T\|^p)^{\frac{1}{p}} < \infty\}$$

is a Banach space w.r.t. $\|\cdot\|_{\mathcal{M}_T^p}$.

Example 1.14. Let $(W(t))_{t\geq 0}$ be a U-valued Q-Wiener process w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Then $(W(t))_{t\geq 0} \in \mathcal{M}_T^2$ with $\mathbb{E}(\|W(t)\|_U^2) = t\,\mathrm{tr}\,(Q)$. Indeed, the independence of the increment $W(t) - W(s)$ of $\mathcal{F}_s$ implies that

$$\mathbb{E}(W(t) - W(s) \mid \mathcal{F}_s) = \mathbb{E}(W(t) - W(s)) = 0$$

and thus

$$\mathbb{E}(W(t) \mid \mathcal{F}_s) = \mathbb{E}(W(t) - W(s) \mid \mathcal{F}_s) + W(s) = W(s) .$$

1.4. **Stochastic integration.** Fix U, H and a U-valued Q-Wiener process $(W(t))_{t\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Because a typical path $s \mapsto W(s)$ is neither differentiable nor of bounded variation, the construction of the stochastic integral $\int_0^t \Phi(s)\, dW(s)$ requires an extension of the classical integration theory. The construction is achieved in four steps.

Step 1: Integration of elementary processes

$$\Phi(t) = \sum_{m=1}^{k-1} \Phi_m 1_{]t_m,t_{m+1}]}(t) \tag{2}$$

where

- $0 = t_0 < t_1 < \ldots < t_k = T$
- $\Phi_m : \Omega \to L(U,H)$ is $\mathcal{F}_{t_m}$-measurable, bounded

Define

$$\int_0^t \Phi(s)\, dW(s) := \sum_{m=0}^{k-1} \Phi_m \left(W(t_{m+1}\wedge t) - W(t_m \wedge t)\right) , 0 \leq t \leq T$$

induces a linear mapping

$$I : \mathcal{E} \to \mathcal{M}_T^2$$

where $\mathcal{E}$ denotes the set of all elementary processes of type (2).

Step 2: Wiener-Ito isometry

Denote by $L_2(U,H)$ the space of all Hilbert-Schmidt operators $L : U \to H$, i.e., $L \in L(U,H)$ and

$$\|L\|_{L_2}^2 := \sum_{k=1}^{\infty} \|Le_k\|_H^2 < \infty$$

for one (hence any) CONS $(e_k)_{k\geq 1}$ of U. Recall that $L_2(U,H)$ is a Hilbert space w.r.t.

$$\langle L, M\rangle_{L_2} = \sum_{k=1}^{\infty} \langle Le_k, Me_k\rangle_H .$$

For $\Phi \in \mathcal{E}$ the following Wiener-Ito isometry holds

(3)

$$\mathbb{E}\left(\left\|\int_0^t \Phi(s)\, dW(s)\right\|_H^2\right) = \mathbb{E}\left(\int_0^t \left\|\Phi(s)\sqrt{Q}\right\|_{L_2(U,H)}^2 \, ds\right) , t \in [0,T] .$$

The right hand side in the last equality can be rewritten in terms of the Hilbert space $U_0 = \sqrt{Q}(U)$ together with the inner product $\langle u_0, v_0\rangle_{U_0} = \langle \sqrt{Q}^{-1} u_0, \sqrt{Q}^{-1} v_0\rangle_U$, where $\sqrt{Q}^{-1}$ denotes the pseudo inverse of $\sqrt{Q}$ if $\sqrt{Q}$ is not one-to-one. Indeed, let $L_2^0 := L_2(U_0, H)$ be the space of all Hilbert-Schmidt operators $T : U_0 \to H$. Then we can rewrite the Wiener-Ito isometry in the form

$$\mathbb{E}\left(\left\|\int_0^t \Phi(s)\, dW(s)\right\|_H^2\right) = \mathbb{E}\left(\int_0^t \|\Phi(s)\|_{L_2^0}^2 \, ds\right) , t \in [0,T] , \tag{4}$$

in particular,

$$\left\|\int_0^{\cdot} \Phi(s)\, dW(s)\right\|_{\mathcal{M}_T^2}^2 = \mathbb{E}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, ds\right) ,$$

hence

$$I : \mathcal{E} \to \mathcal{M}_T^2$$

defines an isometry if $\mathcal{E}$ is endowed with the seminorm

$$\|\Phi\|_T^2 := \mathbb{E}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, ds\right) = \int_\Omega \int_0^T \|\Phi(s)\|_{L_2^0}^2(\omega)\, ds\, \mathbb{P}(d\omega) .$$

It follows that the definition of the stochastic integral I can be extended to integrands contained in the abstract completion $\bar{\mathcal{E}}$ of $\mathcal{E}$ w.r.t. the seminorm $\|\cdot\|_T$.

Step 3: Identification of $\bar{\mathcal{E}}$

To identify the abstract completion of $\mathcal{E}$ let us introduce the following σ-algebra

$$\begin{aligned} \mathcal{P}_T &:= \sigma(\{]s,t] \times F_s \mid 0 \leq s < t \leq T , F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 \mid F_0 \in \mathcal{F}_0\}) \\ &= \sigma(\{(H_t)_{t\in[0,T]} \mid (H_t) \text{ left-continuous}, (\mathcal{F}_t) - \text{adapted}\}) . \end{aligned}$$

$\mathcal{P}_T$ is called the predictable σ-algebra and a $\mathcal{P}_T$-measurable process $(H_t)_{t\in[0,T]}$ is called predictable. Then

$$\bar{\mathcal{E}} := \mathcal{L}^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$$

where

$$\Omega_T = \Omega \times [0,T]\,, \qquad \mathbb{P}_T = \mathbb{P} \otimes dt$$

and L_2^0, as before, denotes the space of all linear operators $L \in L(U,H)$ such that $L \circ \sqrt{Q} \in L_2(U,H)$. Consequently, $I : \mathcal{E} \to \mathcal{M}_T^2$ can be uniquely extended to an isometry

$$\begin{aligned} I : \mathcal{L}^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0) &\to \mathcal{M}_T^2 \\ \Phi(\cdot) &\mapsto \left(\int_0^\cdot \Phi(s)\, dW(s) \right)_{t\in[0,T]} . \end{aligned}$$

Step 4: Localization

Using suitable stopping times, the definition of $\int_0^t \Phi(s)\, dW(s)$, $t \in [0,T]$, can be extended to the space

$$\begin{aligned} \mathcal{N}_W := \{\Phi : \Omega_T \to L_2^0(U,H) \mid \Phi \text{ predictable and} \\ \mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2^0}\, ds < \infty \right) = 1\}\,. \end{aligned}$$

$\mathcal{N}_W$ is called the space of admissible integrands.

Properties of the stochastic integral

(i) Linearity: $L \in L(H, \tilde{H})$ then

$$L\left(\int_0^t \Phi(s)\, dW(s) \right) = \int_0^t L \circ \Phi(s)\, dW(s)$$

(ii) $f : \Omega_T \to H$, $(\mathcal{F}_t)_{t\geq 0}$-adapted, continuous, then

$$\int_0^t \langle f(s), \Phi(s)\, dW(s)\rangle = \int_0^t \tilde{\Phi}_f(s)\, dW(s)\,,$$

where $\tilde{\Phi}_f(s)(u) = \langle f(s), \Phi(s)u\rangle_H$, $u \in H$.

(iii) Let $p \geq 1$. Then there exists a universal constant c_p such that

$$\mathbb{E}\left(\left\| \int_0^t \Phi(s)\, dW(s) \right\|_H^{2p} \right) \leq c_p \mathbb{E}\left(\int_0^t \|\Phi(s)\|^2_{L_2^0}\, ds \right)^p .$$

In particular, the following inequality, called the Burkholder-Davis-Gundy inequality, holds

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\| \int_0^t \Phi(s)\, dW(s) \right\|_H^{2p} \right) \leq c_p \mathbb{E}\left(\int_0^T \|\Phi(s)\|^2_{L_2^0}\, ds \right)^p$$

(iv) Quadratic variation: Let $M_t := \int_0^t \Phi(s)\, dW(s)$, then

$$\langle M\rangle_t := \int_0^t \|\Phi(s)\|^2_{L_2^0}\, ds\,, t \in [0,T]\,,$$

is the unique continuous increasing $(\mathcal{F}_t)_{t\geq 0}$-adapted process starting at zero such that

$$\|M_t\|_H^2 - \langle M\rangle_t\,, t\in[0,T]\,,$$

is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. It can be shown that for any sequence of partitions $(\tau_n)_{n\geq 1}$ of $[0,T]$ with $\lim_{n\to\infty}|\tau_n| = 0$ it follows that

$$\lim_{n\to\infty} \sum_{t_i\in\tau_n, t_i\leq t} \|M_{t_{i+1}} - M_{t_i}\|_H^2 = \langle M\rangle_t \tag{5}$$

uniformly in t, in probability. More general, given any $h \in H$, the process

$$\int_0^t \|\sqrt{Q}\circ\Phi(s)^* h\|_H^2\, ds\,, t\in[0.T],$$

is the unique continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process starting at zero such that

$$\langle M_t, h\rangle_H^2 - \int_0^t \|\sqrt{Q}\circ\Phi(s)^* h\|_H^2\, ds\,, t\in[0,T]\,,$$

is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. In analogy with (5)

$$\lim_{n\to\infty} \sum_{t_i\in\tau_n, t_i\leq t} \big(\langle M_{t_{i+1}}, h\rangle_H - \langle M_{t_i}, h\rangle_H\big)^2 = \int_0^t \|\sqrt{Q}\circ\Phi(s)^* h\|_H^2\, ds$$

uniformly in t, in probability.

(v) Regularity of the stochastic integral: let $\alpha < \frac{1}{2}$ be given. Then for any $\Phi \in \mathcal{L}^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$

$$M_t = \int_0^t \Phi(s)\, dW(s) \in W^{\alpha,2}([0,T]; H)$$

where for a given Banach space E, the space $W^{\alpha,2}([0,T]; E)$ consists of all functions $M \in L^2([0,T]; E)$ satisfying

$$\int_0^T\int_0^T \frac{\|M_s - M_t\|_E^2}{|s-t|^{1+2\alpha}}\, ds\, dt < \infty\,.$$

Proof: The Wiener-Ito isometry implies that

$$\begin{aligned}\mathbb{E}\left(\int_0^T\int_0^T \frac{\|M_t - M_s\|_H^2}{|t-s|^{1+2\alpha}}\,ds\,dt\right) &= \int_0^T\int_0^T \frac{\mathbb{E}\left(\int_{s\wedge t}^{s\vee t}\|\Phi(r)\|_{L_2^0}^2\,dr\right)}{|t-s|^{1+2\alpha}}\,ds\,dt \\ &= 2\int_0^T\int_t^T \frac{\int_t^s \mathbb{E}\left(\|\Phi(r)\|_{L_2^0}^2\right)\,dr}{|t-s|^{1+2\alpha}}\,ds\,dt \\ &\le \ldots \le C(\alpha)\int_0^T \mathbb{E}\left(\|\Phi(r)\|_{L_2^0}^2\right)\,dr\,.\end{aligned}$$

1.5. **Appendix: Stochastic integration w.r.t. cylindrical Wiener processes.** The construction of stochastic integrals $\int_0^t \Phi(s)\,dW(s)$ can be extended to the case where the covariance operator Q is only bounded, but not necessarily of finite trace. To this end, one needs to extend the notion of a Q-Wiener process. To simplify the presentation, we restrict ourselves to the particular case $Q = I$.

The representation of the Q-Wiener process obtained in Theorem 1.8 leads in the case $Q = I$ to the infinite series

$$W(t) = \sum_{k=1}^\infty \beta_k(t)e_k\,, t\in[0,T]\,,$$

for independent one-dimensional Brownian motions β_k, $k\ge 1$. Note that this series does not converge in U, since

$$\left\|\sum_{k=1}^n \beta_k(t)e_k\right\|_U^2 = \sum_{k=1}^n \beta_k(t)^2$$

and thus

$$\mathbb{E}\left(\left\|\sum_{k=1}^n \beta_k(t)e_k\right\|_U^2\right) = \sum_{k=1}^n \mathbb{E}(\beta_k(t)^2) = n\cdot t\uparrow\infty$$

for $n\to\infty$. However, for any Hilbert space $(U_1, \langle\ ,\ \rangle_{U_1})$ for which there exists a Hilbert-Schmidt embedding $J : U \to U_1$ the infinite series converges in U_1, since

$$\left\|J(\sum_{k=1}^n \beta_k(t)e_k)\right\|_{U_1}^2 = \sum_{k=1}^n \beta_k(t)^2\|J(e_k)\|_{U_1}^2$$

and thus

$$\begin{aligned}\mathbb{E}\left(\left\|J(\sum_{k=1}^n \beta_k(t)e_k)\right\|_{U_1}^2\right) &= \sum_{k=1}^n \mathbb{E}(\beta_k(t)^2)\|J(e_k)\|_{U_1}^2 \\ &= \sum_{k=1}^n t\,\|J(e_k)\|_{U_1}^2 \uparrow t\,\|J\|_{L_2(U,U_1)}^2\,.\end{aligned}$$

Remark 1.15. U_1 with the above properties always exists. For example, choose a sequence $(\alpha_k)_{k\geq 1} \in \ell^2$ with $\alpha_k \neq 0$ for all k, let $U_1 = U$ and define

$$J : U \to U_1 \,, u \mapsto \sum_{k=1}^{\infty} \alpha_k \langle u, e_k \rangle_U e_k \,.$$

In the following we fix a sequence of independent one-dimensional Brownian motions β_k, $k \geq 1$, a CONS $(e_k)_{k\geq 1}$ of U and a Hilbert space U_1 for which there exists a Hilbert-Schmidt embedding $J : U \to U_1$. In particular, $Q_1 := J \circ J^* \in L(U_1)$ is symmetric, positive definite with finite trace and the infinite series

$$W_1(t) = \sum_{k=1}^{\infty} \beta_k(t) J(e_k) \,, t \in [0, T] \,,$$

converges in $\mathcal{M}_T^2(U_1)$ and defines a Q_1-Wiener process on U_1.

For a given predictable process Φ satisfying

$$\mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2(U,H)} \, ds < \infty \right) = 1 \,,$$

using

$$\|\Phi(s)\|^2_{L_2(U,H)} = \|\Phi(s) \circ J^{-1}\|^2_{L_2(\sqrt{Q_1}(U_1),H)} \,,$$

we conclude that the stochastic integral

$$\int_0^t \Phi(s) \circ J^{-1} \, dW_1(s)$$

w.r.t. the Q_1-Wiener process is well-defined. Finally, we set

$$\int_0^t \Phi(s) \, dW(s) := \int_0^t \Phi(s) \circ J^{-1} \, dW_1(s) \,.$$

The class of admissible integrands is given by

$$\mathcal{N}_W = \{\Phi : \Omega_T \to L_2(U, H) \mid \Phi \text{ predictable and } \mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2(U,H)} \, ds < \infty \right) = 1\} \,.$$

2. Stochastic Differential Equations on Hilbert spaces

2.1. Mild, weak and strong solutions.

Throughout the whole subsection fix two separable (real) Hilbert spaces U, H and a Q-Wiener process $(W_t)_{t\geq 0}$ w.r.t. $(\mathcal{F}_t)_{t\geq 0}$. Consider the equation

$$\begin{cases} dX_t &= [AX_t + B(X_t)] \, dt + C(X_t) \, dW_t \in H \\ X_0 &= \xi \end{cases} \tag{6}$$

with

(A.1) $(A, D(A))$ generates a C_0-semigroup $(T_t)_{t\geq 0}$ on H

(A.2) $B : H \to H$ is $\mathcal{B}(H)$-measurable

(A.3) $C : H \to L_2(U_0, H)$ is strongly continuous, i.e.,

$$x \mapsto C(x)u\,, H \to H$$

is continuous for all $u \in U_0$.

(A.4) ξ is H-valued and $\mathcal{F}_0$-measurable.

Notions of solutions

- **mild solution:** An H-valued predictable process $(X_t)_{t\in[0,T]}$ satisfying

$$X_t = T_t\xi + \int_0^t T_{t-s}B(X_s)\,ds + \int_0^t T_{t-s}C(X_s)\,dW_s \qquad \mathbb{P}-a.s.\,,$$

  for all $t \in [0,T]$, where all integrals have to be well-defined.
- **(analytically) strong solution:** An $D(A)$-valued predictable process $(X_t)_{t\in[0,T]}$ satisfying

$$X_t = \xi + \int_0^t \left[AX_s + B(X_s)\right]\,ds + \int_0^t C(X_s)\,dW_s \qquad \mathbb{P}-a.s.\,,$$

  for all $t \in [0,T]$, where all integrals have to be well-defined.
- **(analytically) weak solution:** An H-valued predictable process $(X_t)_{t\in[0,T]}$ satisfying

$$\begin{aligned}\langle X_t, \varphi\rangle_H = \langle \xi, \varphi\rangle_H &+ \int_0^t \langle X_s, A^*\varphi\rangle_H + \langle B(X_s), \varphi\rangle_H\,ds \\ &+ \int_0^t \langle \varphi, C(X_s)\,dW_s\rangle_H \qquad \mathbb{P}-a.s.\,,\end{aligned}$$

  for all $t \in [0,T]$, $\varphi \in D(A^*)$. Here, $(A^*, D(A^*))$ is the dual operator of A and it is required that all integrals are well-defined.

The precise interrelations between the three different notions of solutions can be found in [18].

Stochastic differential equations with additive noise

In the particular case where the dispersion coefficient C does not depend on the solution, equation (6) is called a stochastic differential equation with additive noise. This case is very close to the deterministic analogue. Indeed, let

$$W_A(t) := \int_0^t T_{t-s}C\,dW_s\,, \qquad t \in [0,T]\,,$$

be the stochastic convolution and suppose that $(W_A(t))_{t\in[0,T]}$ has a version with continuous trajectories in H. Decomposing the mild solution

$$X_t = Y_t + W_A(t)$$

we formally obtain the following equation

$$dY_t = \left[AY_t + B(Y_t + W_A(t))\right]\,dt\,, Y_0 = \xi \tag{7}$$

for $(Y_t)_{t\in[0,T]}$. Equation (7) can be seen as a deterministic evolution equation with time-dependent random coefficients

$$B(\cdot + W_A(t))\,.$$

In particular, if (7) has a unique mild solution $(Y_t(\omega))_{t\in[0,T]}$ for P-a.e. ω and the dependence of $(Y_t)_{t\in[0,T]}$ on ω is predictable, then $X_t = Y_t + W_A(t)$, $t \in [0,T]$, is a mild solution of (6).

2D-Stochastic Navier Stokes equations with additive noise

Let $D \subset \mathbb{R}^2$ be a bounded open domain with regular boundary ∂D. We consider the following stochastic Navier-Stokes equations

(8)
$$\begin{cases} \partial_t u(t,x) - \nu\Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) &= \dot{\xi}_t(x) \quad t\in[0,T], x\in D \\ \text{div } u(t,x) &= 0 \qquad t\in[0,T], x\in D \\ u(t,x) &= 0 \qquad t\in[0,T], x\in\partial D \\ u(0,x) &= u_0(x) \quad x\in D\,, \end{cases}$$

where $(\xi_t)_{t\ge 0}$ is a (cylindrical) Wiener process. Here, $u : [0,T]\times D \to \mathbb{R}^2$ is the velocity field, $\nu > 0$ the viscosity and $p : [0,T]\times D \to \mathbb{R}$ denotes the pressure. We will consider the equation in similar function spaces as for the deterministic case:

$$\begin{aligned} D_0^\infty &= \left\{u \in C_0^\infty(D;\mathbb{R}^2)\,,\ \text{div } u = 0\right\} \\ H &= \text{ closure of } D_0^\infty \text{ in } L^2(D;\mathbb{R}^2) \text{ w.r.t. } \|u\|_H^2 := \int_D |u|^2\,dx \\ V &= \text{ closure of } D_0^\infty \text{ in } L^2(D;\mathbb{R}^2) \text{ w.r.t. } \|u\|_V^2 := \int_D |Du|^2\,dx \end{aligned}$$

Applying the Helmholtz projection $\Pi : L^2(D,\mathbb{R}^2) \to H$ one obtains the following abstract evolution equation

$$\text{(9)} \qquad \begin{cases} du(t) &= [Au(t) + B(u(t),u(t)) + f(t)]\,dt + C\,dW_t \\ u(0) &= u_0 \end{cases}$$

on H, where

- $A = \Pi\Delta_D$ is the Stokes operator on H
- $B : V\times V \to V'$, ${}_{V'}\langle B(u,v),w\rangle_V = -\int_D w(x)\cdot(u(x)\cdot\nabla)v(x)\,dx$.

Theorem 2.1. *If* $(W_A(t))_{t\in[0,T]}$ *has a version in* V *with continuous trajectories, then* (9) *has a unique mild solution.*

Proof. In this case, equation (7) can be written as

$$\text{(10)} \qquad dY_t = [\nu AY_t + B(Y_t + W_A(t))]\,dt\,, Y_0 = u_0\,.$$

It is a classical result, that (10) has for ω with $t \mapsto W_A(t)(\omega)$, $[0,T] \to V$ continuous, a unique solution $Y_\cdot \in L^2([0,T];V)$, $\dot{Y}_\cdot \in L^1([0,T];V')$

satisfying also

$$\sup_{t\in[0,T]} \|Y_t\|_H^2 + \nu \int_0^T \|Y_t\|_V^2 \, dt \\ \le \exp\left(\frac{2}{\nu}\int_0^T \|W_A(t)(\omega)\|_V^2 \, dt\right) \\ \left(\|u_0\|_H^2 + \frac{1}{\nu}\int_0^T \|W_A(t)(\omega)\|_H^2 \|W_A(t)(\omega)\|_V^2 \, dt\right) .$$

It can be also shown that the dependence of the unique solution $Y_\cdot$ on ω is predictable, so that $X_t = Y_t + W_A(t)$ is a mild solution of (9). $\square$

Remark 2.2. Regularity properties of the stochastic convolution W_A are well-studied (see the monographs [3, 4]). The main difficulty with W_A is that it is not a martingale w.r.t. t. This does not contradict the properties of the stochastic integral, because for any $t > 0$ the process

$$W_A^{(t)}(s) = \int_0^s e^{(t-r)A} C \, dW_r \, , s \in [0,t] \, ,$$

is a martingale up to time t.

2.2. **Existence and uniqueness of mild solutions.** In this subsection we discuss existence and uniqueness of mild solutions of (6) under the following additional assumption:

$$\text{(H.1)} \qquad \|B(x)-B(y)\|_H + \|C(x)-C(y)\|_{L_2^0} \le M\|x-y\|_H \quad \forall x,y \in H$$

Theorem 2.3. *Under hypotheses (A.1)-(A.4), (H.1) there exists a unique mild solution of* (6) *satisfying*

$$\mathbb{P}\left(\int_0^T \|X_s\|_H^2 \, ds < \infty\right) = 1 \, .$$

$(X_t)_{t\in[0,T]}$ *has a continuous modification and*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^p\right) \le c_{p,T}\left(1 + \mathbb{E}\left(\|\xi\|^p\right)\right) \qquad \forall p > 2 \, .$$

We give a sketch of the proof of existence of a mild solution in the case $\mathbb{E}\left(\|\xi\|_H^2\right) < \infty$. The basic ingredient is provided by Banach's fixed point theorem applied to the space

$$\mathcal{H}_p := \{Y : \Omega_T \to H \mid Y \text{ predictable }, |Y|_p := \sup_{t\in[0,T]} \mathbb{E}\left(\|Y_t\|^p\right)^{\frac{1}{p}} < \infty\}$$

and the mapping

$$\mathcal{K}(Y)(t) := T_t\xi + \int_0^t T_{t-s} B(Y_s) \, ds + \int_0^t T_{t-s} C(Y_s) \, dW_s \\ = T_t\xi + \mathcal{K}_1(Y)(t) + \mathcal{K}_2(Y)(t) \, , \text{ say.}$$

Then $\mathbb{E}\left(\|\xi\|^p\right) < \infty$ implies that $\mathcal{K}(\mathcal{H}_p) \subset \mathcal{H}_p$ and is a strict contraction for small T. Indeed, for $0 \le t \le T$

$$\mathbb{E}\left(\|\mathcal{K}_1(Y)(t)\|^p\right) \le \mathbb{E}\left(\left(\int_0^t \|B(Y_s)\|\,ds\right)^p\right) \le c_p T^p \left(1 + |Y|_p^p\right)$$

and the Burkholder-Davis-Gundy inequality implies that

$$\begin{aligned}\mathbb{E}\left(\|\mathcal{K}_2(Y)(t)\|^p\right) &= \mathbb{E}\left(\|\int_0^t T_{t-s}C(Y_s)\,dW_s\|^p\right)\\ &\le c_{\frac{p}{2}}\mathbb{E}\left(\int_0^t \|T_{t-s}C(Y_s)\|^2_{L_2^0}\,ds\right)^{\frac{p}{2}}\\ &\le \ldots \le c_p T^{\frac{p}{2}}\left(1 + |Y|_p^p\right).\end{aligned}$$

Similarly,

$$\mathbb{E}\left(\|\mathcal{K}(Y_1)(t) - \mathcal{K}(Y_2)(t)\|^p\right) \le c_p T^{\frac{p}{2}}\left(T^{\frac{p}{2}} + 1\right)|Y_1 - Y_2|_p^p$$

and then proceed as expected. □

2.3. **Martingale solutions and basic existence theorem.** The concept of mild solutions is essentially restricted to stochastic partial differential equations with one-sided Lipschitz continuous drift B and Lipschitz continuous dispersion coefficient $C(\cdot) : H \to L_2(U_0, H)$. To cover more singular equations one needs to weaken the notion of a solution. To this end suppose that (X_t) is a weak solution of (6). Note that in this case for all $\varphi \in D(A^*)$

$$\begin{aligned}M_t^\varphi &:= \langle X_t, \varphi\rangle_H - \langle \xi, \varphi\rangle_H - \int_0^t \langle X_s, A^*\varphi\rangle_H + \langle B(X_s), \varphi\rangle_H\,ds\\ &= \int_0^t \langle \varphi, C(X_s)\,dW_s\rangle_H\,, 0 \le t \le T\end{aligned}$$

is an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale with quadratic variation

$$\langle M^\varphi\rangle_t = \int_0^t \|\sqrt{Q} \circ C^*(X_s)\varphi\|_U^2\,ds\,.$$

to simplify the presentation in the following we make the assumptions:

(B.1) $(A, D(A))$ is self-adjoint, $\langle Au, u\rangle_H \le 0$ for all $u \in D(A)$.

In the following, let $V := D(\sqrt{-A})$, equipped with the scalar product

$$\langle u, v\rangle_V := \langle \sqrt{-A}u, \sqrt{-A}v\rangle_H + \langle u, v\rangle_H\,.$$

Identifying H with its dual H', we obtain the following continuous and dense embeddings

$$D(A) \hookrightarrow V \hookrightarrow H \cong H' \hookrightarrow V' \hookrightarrow D(A)'\,.$$

(B.2) $B : V \to V'$ is continuous, $\exists \gamma \ge 0$ such that $\|B(u)\|_{D((-A)^\gamma)'} \le M\left(1 + \|u\|_H\|u\|_V\right)$

(B.3) $C : V \to L(U, H)$ is continuous
(B.4) μ_0 is a probability measure on H having finite second moments

$$\int_H \|x\|_H^2 \, \mu_0(dx) < \infty \,.$$

Definition 2.4. **(martingale solution)**
A probability measure $\mathbb{P}$ on $\Omega = L^2([0,T];V)$ for which the canonical process $X_t : \Omega \to V$, $\omega \mapsto \omega(t)$, satisfies

(i)

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|X_t\|_H^2 + \int_0^T \|X_t\|_V^2 \, dt < \infty\right) = 1$$

(ii)

$$\begin{aligned} M_t^\varphi &:= \langle X_t, \varphi\rangle_H - \langle \xi, \varphi\rangle_H \\ &\quad - \int_0^t \langle X_s, A\varphi\rangle_H + {}_{V'}\langle B(X_s), \varphi\rangle_V \, ds \,, 0 \le t \le T \,, \end{aligned}$$

is a martingale with

$$\langle M^\varphi\rangle_t = \int_0^t \|\sqrt{Q} \circ C^*(X_s)\varphi\|_U^2 \, ds$$

for all $\varphi \in D(A)$

(iii) $\mathbb{P} \circ X_0^{-1} = \mu_0$

is called a martingale solution of (6) with initial condition μ_0.

We will prove the existence of a martingale solution of (6) under the following additional assumption:

(H.2) (i) $V \hookrightarrow H$ is compact
(ii) $\exists\, \eta \in]0, 2]$, λ_0 and ρ such that

$$2\langle Au + B(u), u\rangle_H + \|C(u)\|_{L_2(U_0,H)}^2 \le -\eta \|u\|_V^2 + \lambda_0 \|u\|_H^2 + \rho \qquad \forall u \in V$$

(iii) there exists a dense subset $V_0 \subset V$ such that for all $v \in V_0$ the mappings

$$\begin{aligned} &u \mapsto \langle B(u), v\rangle \,, V \to \mathbb{R} \\ &u \mapsto C(u)^* v \,, V \to U_0 \end{aligned}$$

can be extended by continuity to continuous mappings $H \to \mathbb{R}$ and $H \to U_0$ and in addition

$$\|C(u)^* v\|_{U_0}^2 \le c(v)\left(1 + \|u\|_H^2\right) \,.$$

Theorem 2.5. *Under hypotheses (B.1)-(B.4), (H.2) there exists a martingale solution of* (6) *satisfying*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^2 + \int_0^T \|X_t\|_V^2 \, dt\right) < \infty \,.$$

In addition,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^p\right) < \infty$$

for any $p \geq 2$ *with* $\int \|x\|_H^p \, \mu_0(dx) < \infty$.

Proof: (Sketch) basic ingredient: compactness of the laws $\mathbb{P}^n$ of finite dimensional Galerkin approximations $(X_t^n)_{t\geq 0}$ of (6). The necessary mathematical background needed in the proof is provided in the appendix to this section.

Step 1: Finite dimensional approximations

Let $(e_k)_{k\geq 1}$ be a CONS of H, consisting of eigenvectors of A, let $\pi_n : H \to H^n$, $x \mapsto \sum_{k=1}^n \langle x, e_k\rangle_H e_k$ be the canonical projection on the linear span of the first n eigenvectors, and let $B_n(x) := \pi_n B(x)$, $C_n(x) := \pi_n C(x)$. We then consider the stochastic differential equation

$$dX_t^n = [AX_t^n + B_n(X_t^n)]\, dt + C_n(X_t^n)\, dW_t$$

on the finite dimensional space H^n, and let $\mu_0^n := \mu_0 \circ \pi_n^{-1}$. (H.2) implies for all $u \in H^n \subset D(A)$

$$2\langle Au + B_n(u), u\rangle_H + \|C_n(u)\|^2_{L_2(U_0,H)} \leq -\eta\|u\|_V^2 + \lambda_0\|u\|_H^2 + \rho$$

with η, λ_0 and ρ as in (H.2), in particular uniformly in n.

Standard results for stochastic differential equations imply the existence of a martingale solution

$$X_\cdot^n \in L^2(\Omega, C([0,T]; H^n))$$

together with the moment estimate

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t^n\|_H^2 + \int_0^T \|X_t^n\|_V^2 \, dt\right) \leq C_1 \tag{11}$$

uniformly in n and in addition

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t^n\|_H^p\right) \leq C_2(p) \tag{12}$$

uniformly in n for any $p > 2$ with $\int_H \|x\|_H^p \, \mu_0(dx) < \infty$.

Step 2: Tightness of $\mathbb{P}^n := \mathbb{P} \circ (X_\cdot^n)^{-1}$, $n \geq 1$, on $L^2([0,T]; H)$

Recall the definition of the space $W^{\alpha,2}([0,T]; E)$ for a Banach space E. Since $V \hookrightarrow H$ is compact, we obtain that

$$L^2([0,T]; V) \cap W^{\alpha,2}([0,T]; D((-A)^\gamma)') \hookrightarrow L^2([0,T]; H)$$

is compact.Therefore it suffices to prove that $X^n_\cdot$ is bounded in $L^2([0,T];V)\cap W^{\alpha,2}([0,T];D((-A)^\gamma)')$ in probability. To this end, let us decompose the solution

$$\begin{aligned} X^n_t &= X^n_0 + \int_0^t AX^n_s\,ds + \int_0^t B_n(X^n_s)\,ds + \int_0^t C_n(X^n_s)\,dW_s \\ &= I^n_1 + I^n_2(t) + I^n_3(t) + I^n_4(t) \qquad \text{say.} \end{aligned}$$

Then

$$\mathbb{E}\left(\|I^n_1\|^2_H\right) \le C_1 \quad \text{and} \quad \mathbb{E}\left(\|I^n_2\|^2_{W^{1,2}([0,T];V')}\right) \le C_2$$

for uniform constants. In addition,

$$\mathbb{E}\left(\|I^n_4\|^2_{W^{\alpha,2}([0,T];H)}\right) \le C_4(\alpha)$$

for all $\alpha \in (0,\frac{1}{2})$, and by assumption on B

$$\mathbb{E}\left(\|B^n(X^n_\cdot)\|_{L^2([0,T];D((-A)^\gamma)')}\right) \le M\left(1+\mathbb{E}\left(\sup_{t\in[0,T]}\|X^n(t)\|^2_H + \int_0^T \|X^n_t\|^2_V\,dt\right)\right)$$

is uniformly bounded in n, hence

$$\mathbb{E}\left(\|I^n_3\|_{W^{1,2}([0,T];D((-A)^\gamma)')}\right) \le C_3\,.$$

Combining the estimates, we conclude that

$$\mathbb{E}\left(\|X^n_\cdot\|_{W^{\alpha,2}([0,T];D((-A)^\gamma)')}\right) \le C(\alpha) \tag{13}$$

for some constant $C(\alpha)$ uniform in n, hence the assertion follows.

Step 3: Identification of the limit measures

Let $\mathbb{P}^\infty$ be the limit of some weakly convergent subsequence of $\mathbb{P}^n$ on $L^2([0,T];H)$, again denoted by $\mathbb{P}^n$, $n\ge 1$. The Skorohod embedding theorem implies the existence of some probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ with filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ and a sequence of stochastic processes $\tilde X^n$, $\tilde X \in L^2([0,T];H)$ such that

$$\tilde X^n_\cdot \to \tilde X_\cdot \quad \tilde{\mathbb{P}}-\text{a.s. in } C([0,T];D((-A)^\gamma)')\,.$$

Clearly, $\tilde X^n$, and consequently $\tilde X$ too, satisfy the moment estimates (11) and (12). This implies that $\tilde X^n \to \tilde X$ weakly in $L^2([0,T];V)$ $\tilde{\mathbb{P}}$-a.s. and strongly in $L^2([0,T];H)$ $\tilde{\mathbb{P}}$-a.s. For all n

$$\tilde M^n_t = \tilde X^n_t - \tilde X^n_0 - \int_0^t A\tilde X^n_s + B(\tilde X^n_s)\,ds$$

is a martingale w.r.t. to $\tilde{\mathcal{G}}^n_t := \sigma\left(\tilde X^n_s \mid s\le t\right)$ with quadratic variation

$$\langle \tilde M^n\rangle_t = \int_0^t C_n(\tilde X^n_s)C_n(\tilde X^n_s)^*\,ds\,.$$

(H.2) implies that

$$\langle \int_0^t B_n(\tilde{X}_s^n)\, ds, v\rangle \to \langle \int_0^t B(\tilde{X}_s)\, ds, v\rangle \qquad \forall v \in V_0$$

and

$$\int_0^t \|C_n(\tilde{X}_s^n)^* v\|_{U_0}^2\, ds \to \int_0^t \|C(\tilde{X}_s)^* v\|_{U_0}^2\, ds \qquad \forall v \in V_0\,.$$

Hence

$$\tilde{M}_t = \tilde{X}_t - \tilde{X}_0 - \int_0^t A\tilde{X}_s + B(\tilde{X}_s)\, ds$$

is a martingale w.r.t. to $\tilde{\mathcal{G}}_t := \sigma\left(\tilde{X}_s \mid s \le t\right)$ with quadratic variation

$$\langle \tilde{M}\rangle_t = \int_0^t \sqrt{Q} C(\tilde{X}_s) C(\tilde{X}_s)^* \sqrt{Q}\, ds\,.$$

The martingale representation theorem now implies the existence of a Q-Wiener process $(\tilde{W}_t)$ on (some extension of) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\tilde{M}_t = \int_0^t C(\tilde{X}_s)\, d\tilde{W}_s\,. \quad \square$$

2.4. **Appendix: Additional background from stochastic analysis.** The construction of martingale solutions of (6) is based on a compactness method. The purpose of this appendix is to shortly summarize the necessary tools from stochastic analysis.

(A) Relative compactness and tightness of probability measures

Throughout the whole appendix, (S, d) denotes a complete separable metric space. The space $C([0,T]; S)$ of all continuous mappings $x : [0,T] \to S$ is again a complete separable metric space w.r.t. the uniform metric

$$\hat{d}(x, y) := \sup_{t\in[0,T]} d(x_t, y_t)\,.$$

It is easy to see that

$$\mathcal{B}(C([0,T]; S)) = \sigma(\pi_t \mid t \in [0,T])\,,$$

where $\pi_t : C([0,T]; S) \to S$, $x \mapsto x_t$, is the usual evaluation map.

Definition 2.6. A sequence of probability measures $(\mu_n)_{n\ge 1}$ on a metric space E is said to converge weakly to some probability measure μ on S, if

$$\lim_{n\to\infty} \int_E f\, d\mu_n = \int_E f\, d\mu \qquad \forall f \in C_b(E)\,.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X^n : \Omega \to C([0,T];S)$, $n \geq 1$, and $X : \Omega \to C([0,T];S)$ be S-valued stochastic processes. If the sequence $\mu_n := \mathbb{P} \circ (X^n)^{-1}$, $n \geq 1$, of distributions converges weakly to the distribution $\mu := \mathbb{P} \circ X^{-1}$ then in particular all finite dimensional distributions of X^n converge weakly to the finite dimensional distribution of X, i.e., for all $0 \leq t_1 < \ldots < t_k \leq T$,

$$\lim_{n\to\infty} \mathbb{P} \circ (X^n_{t_1}, \ldots, X^n_{t_k})^{-1} = \mathbb{P} \circ (X_{t_1}, \ldots, X_{t_k})^{-1}$$

weakly. The converse is not true in general, in particular, the convergence of the finite dimensional distributions of some stochastic process does not imply the relative compactness of its distributions. Instead, the relative compactness must be assumed in addition. The following theorem by Prohorov is the key for a simple characterization of sequences of probability measures that are relatively compact (w.r.t. the topology of weak convergence).

Theorem 2.7. *(Prohorov's theorem)*
Let E be a complete separable metric space and ξ^n, $n \geq 1$, be a sequence of of E-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following statements are equivalent:

(i) $\mu_n := \mathbb{P} \circ (\xi^n)^{-1}$, $n \geq 1$, *is relatively compact.*
(ii) *The sequence ξ^n, $n \geq 1$ is tight, i.e., for all $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset E$ such that*

$$\mathbb{P}(\xi^n \in K_\varepsilon) \geq 1 - \varepsilon \qquad \forall n \geq 1\,.$$

Proof: Theorem 14.3 in [15].

The weak convergence of finite dimensional distributions together with the tightness of a sequence of stochastic processes yields the most widely used criterion for the distributional convergence of stochastic processes: let $(X^n_t)_{t\in[0,T]}$, $n \geq 1$, and $(X_t)_{t\in[0,T]}$ be S-valued stochastic processes. Then $\lim_{n\to\infty} X^n = X$ in distribution, if and only if the following two conditions hold:

(i) the finite dimensional distributions of X^n, $n \geq 1$, converge weakly to the finite dimensional distributions of X.
(ii) The sequence of distributions of X^n, $n \geq 1$, is tight.

Sufficient conditions for tightness of stochastic processes X^n in $C([0,T];S)$ are based on the Arzela-Ascoli characterization of relatively compact subsets of $C([0,T];S)$.

Theorem 2.8. *(Tightness in $C([0,T];S)$)*

A sequence of S-valued stochastic processes X^n, $n \geq 1$, defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is tight if and only if

$$\lim_{R\to\infty} \sup_{n\geq 1} \mathbb{P}\left(d(X_0^n, o) \geq R\right) = 0 \quad \textit{for some point } o \in S$$

$$\lim_{h\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{s,t\in[0,T],|s-t|\leq h} d(X_s^n, X_t^n) > \varepsilon \right) = 0 \qquad \forall \varepsilon > 0 .$$

Proof: Theorem 14.5 in [15].

It is sometimes much easier to characterize tightness using known compact embeddings for Banach spaces. The following compactness result from [9] is used in the proof of Theorem 2.5.

Theorem 2.9. *Let $E_0 \subset E \subset E_1$ be Banach spaces, E_0 and E_1 reflexive, with compact embedding of E_0 in E. Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$ be given. Then the space*

$$L^p([0,T]; E_0) \cap W^{\alpha,p}([0,T]; E_1)$$

endowed with the usual norm is compactly embedded in $L^p([0,T]; E)$. Here, $W^{\alpha,p}([0,T]; E_1)$ denotes the Sobolev space of all $u \in L^p([0,T]; E_1)$ such that

$$\int_0^T \int_0^T \frac{\|u(t) - u(s)\|_{E_1}^p}{|t-s|^{1+\alpha p}} \, dt \, ds < \infty$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}([0,T];E_1)}^p = \int_0^T \|u(t)\|_{E_1}^p \, dt + \int_0^T \int_0^T \frac{\|u(t) - u(s)\|_{E_1}^p}{|t-s|^{1+\alpha p}} \, dt \, ds .$$

Corollary 2.10. *Let $(X_t^n)_{t\in[0,T]}$, $n \geq 1$, be a family of E_1-valued stochastic processes on some underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$. If $X^n \subset L^p([0,T]; E_0) \cap W^{\alpha,p}([0,T]; E_1)$ is bounded in probability, i.e.,*

$$\lim_{K\to\infty} \sup_{n\geq 1} \mathbb{P}\left(\|X^n\|_{L^p([0,T];E_0)\cap W^{\alpha,p}([0,T];E_1)} \geq K \right) = 0$$

then $\mathbb{P} \circ (X^n)^{-1}$, $n \geq 1$, is tight on $L^2([0,T]; E)$.

(B) Skorohod's embedding theorem turning convergence in distribution into pointwise convergence

Theorem 2.11. *Let E be a complete separable metric spaces and μ_n, $n \geq 1$, μ be distributions on E such that $\lim_{n\to\infty} \mu_n = \mu$ weakly. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and E-valued random variables X^n, $n \geq 1$, and X such that*

(i) $\mathbb{P} \circ (X^n)^{-1} = \mu_n$, $n \geq 1$, *and* $\mathbb{P} \circ X^{-1} = \mu$

(ii) $\lim_{n\to\infty} X^n = X$ $\mathbb{P}$-*a.s.*

Proof: Theorem I.2.7 in [14].

(C) Representation theorem for continuous square integrable martingales

Recall that the stochastic integral $\int_0^t \Phi(s)\, dW_s$, $t \in [0,T]$, w.r.t. some Q-Wiener process is a continuous (local) martingale. The converse statement, contained in the following theorem, is called the representation theorem for continuous martingales:

Theorem 2.12. *Let $M \in \mathcal{M}_T^2$ and*

$$\langle M\rangle_t = \int_0^t \Phi(s) \circ Q \circ \Phi(s)^*\, ds\,, t \in [0,T]$$

where $\Phi \in L_2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T)$ and $Q \in L_2(U)$ is symmetric and positive semidefinite. Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and a U-valued Q-Wiener process W, defined on $(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, \mathbb{P} \otimes \tilde{\mathbb{P}})$, adapted to $(\mathcal{F}_t \otimes \tilde{\mathcal{F}}_t)_{t\geq 0}$, such that

$$M_t(\omega, \tilde{\omega}) = \int_0^t \Phi(s, \omega)\, dW_s(\omega, \tilde{\omega})\,, t \in [0,T]\,.$$

Proof: Theorem 8.2 in [3].

3. Stochastic Navier-Stokes Equations

3.1. **Basic existence result.** Throughout this subsection let $D \subset \mathbb{R}^3$ be a bounded, open domain with regular boundary ∂D. We consider the following stochastic Navier-Stokes equations

(14)
$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + (u(t,x)\cdot\nabla)u(t,x) + \nabla p(t,x) &= f(t,x) + C(u,\xi)(t,x) \quad t \in [0,T], x \in D \\ \text{div } u(t,x) &= 0 \quad t \in [0,T], x \in D \\ u(t,x) &= 0 \quad t \in [0,T], x \in \partial D \\ u(0,x) &= u_0(x) \quad x \in D\,, \end{cases}$$

where ξ is a (cylindrical) Wiener process. The relevant function spaces are the same as for the 2D-case:

$$D_0^\infty = \left\{ u \in C_0^\infty(D; \mathbb{R}^3)\,,\ \text{div } u = 0 \right\}$$

$$H = \text{ closure of } D_0^\infty \text{ in } L^2(D;\mathbb{R}^3) \text{ w.r.t. } \|u\|_H^2 := \int_D |u|^2\, dx$$

$$V = \text{ closure of } D_0^\infty \text{ in } L^2(D;\mathbb{R}^3) \text{ w.r.t. } \|u\|_V^2 := \int_D |Du|^2\, dx$$

Applying the Helmholtz projection $\Pi : L^2(D;\mathbb{R}^3) \to H$ one obtains the following abstract evolution equation

$$\begin{cases} du(t) &= [Au(t) + B(u(t), u(t)) + f(t)]\ dt + C(u(t))\, dW_t \\ u(0) &= u_0 \end{cases} \tag{15}$$

on H, where

- $A = \Pi\Delta_D$ is the Stokes operator on H

- $B : V \times V \to V'$, ${}_{V'}\langle B(u,v), w\rangle_V = -\int_D w(x)\cdot(u(x)\cdot\nabla)v(x)\,dx$.

We make the following assumptions on f and (W_t):

(C.1) $f \in L^2(0,T;V')$

(C.2) (W_t) is a cylindrical Wiener process on some Hilbert space U.

Concerning the dispersion coefficient C we either assume that

(C.3)

$$C(u)h := \sum_{k=1}^{\infty} \sigma_k(u)\langle h, e_k\rangle_U e_k$$

for some ONS $(e_k)_{k\geq 1}$ of H and with $\sigma_k : H \to \mathbb{R}$ equicontinuous and

$$\sum_{k=1}^{\infty} \|\sigma_k\|_\infty^2 \leq \lambda_0 \|u\|_H^2 + \rho$$

or

(C.3)'

$$C(u,\xi)(t,x) = \sum_{k=1}^{N} \left[\langle (b^k(x)\cdot\nabla)u(t,x) + C^k(x)u(t,x)\right]\, d\beta_k(t)$$

where (β_k), $k = 1,\dots,N$, are independent one-dimensional Brownian motions, $b^1,\dots,b^N \in C^\infty(\bar{D} : \mathbb{R}^3)$, $c^1,\dots,c^N \in C^\infty(\bar{D};\mathbb{R})$, such that

$$\sum_{j,k=1}^{3}\left(2\delta_{jk} - \sum_{i=1}^{N} b_j^i(x) b_k^i(x)\right) h_i h_j \geq \eta |h|^2\,, h \in \mathbb{R}^3\,.$$

In this case, $U = \mathbb{R}^N$, is finite dimensional and

$$C(u)h = \Pi\left(\sum_{k=1}^{N}\left(b^k\cdot\nabla\right)u + c^k u\right) h_k\,.$$

As a consequence of Theorem 2.5 we now obtain the following result.

Theorem 3.1. *Under the above assumptions (C.1)-(C.2) and (C.3) (resp. (C.3)') there exists a martingale solution to* (15).

3.2. **Stationary martingale solutions and invariant measures.** A stochastic process $(X_t)_{t\geq 0}$ is called stationary it the distribution of the time-shifted process $(X_{s+\cdot})_{t\geq 0}$ is independent of s, i.e.,

$$\mathbb{P}\circ(X_{s+\cdot})^{-1} = \mathbb{P}\circ(X_\cdot)^{-1}, \qquad s \geq 0\,.$$

Stationarity implies in particular that the distribution of X_t is independent of t. Therefore, $\mu = \mathbb{P}\circ X_0^{-1}$ is called an invariant (probability) measure for $(X_t)_{t\geq 0}$.

A particular example for a stationary process is a Markov process $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ on a state space $(S, \mathcal{S})$ having an invariant measure μ in the sense that for any $\mathcal{E}$-measurable bounded function F

$$\int E_x(F(X_t))\mu(dx) = \int F(x)\mu(dx), \qquad t \geq 0\,. \tag{16}$$

In this case, the distribution of the time-shifted process $(X_{s+t})_{t\geq 0}$ with respect to the probability measure

$$\mathbb{P}_\mu(A) := \int \mathbb{P}_x(A)\mu(dx)$$

will be independent of s.

In the case of the stochastic Navier-Stokes equations we are yet far away from the construction of a full Markov process. For recent results concerning the construction of Markov selections, however, see the paper [11] by Flandoli and Romito. Note that, using the same techniques as in Section 2, we are able to construct a stationary martingale solution under some additional assumption on the coefficients.

Theorem 3.2. *Assume that $f \in V'$ is constant. Under the hypotheses (C.3) (resp. (C.3)') and $\eta\lambda_1 > \lambda_0$ there exists a stationary martingale solution of* (15). *Here, λ_1 denotes the lowest eigenvalue of $-A$.*

Remark 3.3. The existence of invariant measures for stochastic partial differential equations, as well as moment estimates and support properties, are of particular interest, since it opens the door for a study of the associated Fokker-Planck equation satisfied by the associated transition probabilities of the solution. Clearly, this is beyond the scope of this short course, instead we refer to the papers [5, 6, 7, 20, 13].

3.3. **(Analytically) Strong solutions to 3D-stochastic Navier-Stokes equations.** In this subsection we study (analytically) strong solutions of (15) for the particular case $f = 0$. Let us introduce the interpolation spaces

$$H^s := D\left((-A)^{\frac{s}{2}}\right), s \in \mathbb{R}\,,$$

equipped with the norm $\|u\|_{H^s} := \|(-A)^{\frac{s}{2}}u\|_H$. We will prove local existence resp. uniqueness of strong solutions under the following assumption (S.1) resp. (S.2):

(S.1) $C \in C\left(H, L_2(U, H^\delta)\right)$ for some $\delta > 1$ with $\|C(u)\|_{L_2(U,H^\delta)} \leq C(1 + \|u\|_H)$

(S.2) $C \in C\left(H, L_2(U, H^\delta)\right)$ for some $\delta > 1$ with $\|C(u)-C(v)\|_{L_2(U,H^\delta)} \leq C\left(1 + \|u - v\|_H\right)$

To formulate precisely our results on local strong existence and uniqueness we need the following preparations:

Definition 3.4. A martingale solution X of (15) satisfies the **energy inequality** if for all $p \in [2, \infty[$ and all $t \in [0, T]$

$$\begin{aligned}&\mathbb{E}\left(\|X_t\|_H^P + p\int_0^t \|X_s\|_V^2 \|X_2\|_H^{p-2}\, ds\right)\\ &\quad\leq \mathbb{E}\left(\|X_0\|_H^P + \frac{p(p-1)}{2}\int_0^t \|C(X_s)\|_{L_2(U,H)}^2 \|X_s\|_H^{p-2}\, ds\right) \qquad \square\end{aligned}$$

A key role in the following analysis will be played by the following

$$\Phi^X(t) := \int_0^t e^{(t-s)A}C(X_s)dW_s, \quad t \in [D, T]$$

Lemma 3.5. *Assume (S.1) and let X be a martingale solution to* (15). *Then $\Phi^X \in C([0,T]; H^{\frac{5}{4}})$.*

Proof: Let $\alpha \in \left(0, \frac{1}{2}\right)$ and $q \in (1,2)$ be such that $q\left(\alpha - \frac{9}{8}\right) > -1$. The factorization lemma allows to rewrite the stochastic convolution as

$$\Phi^X(t) = \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-s)^{\alpha-1}\, e^{(t-s)A}\xi(s)\, ds\,,$$

where

$$\xi(s) = \int_0^s (s-r)^{-\alpha}\, e^{(s-r)A}\, C(X(r))dW_r\,.$$

We know that

$$\begin{aligned}\|\Phi^X(t)\|_{H^{\frac{5}{4}}} &\leq \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-s)^{\alpha-1}\|e^{(t-s)A}\xi(s)\|_{H^{\frac{5}{4}}}\, ds\\ &\leq \frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-s)^{\alpha-1}\|A^{\frac{1}{2}\left(\frac{5}{4}-1\right)}e^{(t-s)A}\|_{L(H,H)}\|\xi(s)\|_V\, ds\\ &\leq C_1\frac{\sin(\alpha\pi)}{\pi}\int_0^t (t-s)^{\alpha-\frac{9}{8}}\,\|\xi(s)\|_V\, ds\\ &\leq C_1\frac{\sin(\alpha\pi)}{\pi}\left(\int_0^t (t-s)^{q\left(\alpha-\frac{9}{8}\right)}\, ds\right)^{\frac{1}{q}}\cdot\left(\int_0^t \|\xi(s)\|_V^{q^*}\, ds\right)^{\frac{1}{q^*}}.\end{aligned}$$

Using well-known smoothing properties of convolutions (see [4], Appendix A), it suffices now to show that $\int_0^t \|\xi(s)\|_V^{q^*}\, ds < \infty$ $\mathbb{P}$-a.s. To this end note that

$$
\begin{aligned}
\mathbb{E}\left(\|\xi(s)\|_V^{q^*}\right) &= \mathbb{E}\left(\|\int_0^s (s-r)^{-\alpha} e^{(s-r)A} C(X(r))dW_r\|_V^{q^*}\right) \\
&= \mathbb{E}\left(\|\int_0^s (s-r)^{-\alpha}(-A)^{\frac{1}{2}} e^{(s-r)A} C(X(r))dW_r\|_H^{q^*}\right) \\
&\le C_{\frac{q^*}{2}} \mathbb{E}\left(\int_0^s \|(s-r)^{-\alpha}(-A)^{\frac{1}{2}} e^{(s-r)A} C(X(r))\|_{L_2}^2 \, ds\right)^{\frac{q^*}{2}} \\
&\le C_{\frac{q^*}{2}} \mathbb{E}\left(\int_0^s (s-r)^{-2\alpha} C_1^2 (1+\|X(r)\|_H^2)\, dr\right) \\
&\le C_{\frac{q^*}{2}} \left(\int_0^s (s-r)^{-2\alpha\gamma} dr\right)^{\frac{q^*}{2\gamma}} \cdot \mathbb{E}\left(\int_0^s (1+\|X(r)\|_H^2)^{\gamma^*} dr\right)^{\frac{q^*}{2\gamma^*}}
\end{aligned}
$$

where $\gamma > \frac{1}{2}$ is such that $2\alpha\gamma < 1$. $\square$

In the following let

$$
\tau_M(r,\varphi) := \sup\left\{ t \in [0,T] : \int_0^t (1+|\varphi(s)|_{H^{\frac{5}{4}}}^4)\, ds \le \frac{M}{(1+r^2)^2} \right\}.
$$

The previous lemma implies that $\tau_M(\|X_0\|_V, \Phi^X)$ is well-defined for $X(0) \in V$ and an $(\mathcal{F}_t)_{t\ge 0}$-stopping time.

Theorem 3.6. *(see* [1]*) Assume (S.1). Then there exists M such that for each $X_0 \in V$ there exists a strong solution on the (random) time interval* $[0, \tau_M(\|X_0\|_V, \Phi^X)[$.

Remark 3.7. It can be further shown that for each $p \in [1,\infty[$ there exists a constant $N(p)$ such that for all martingale solutions X that satisfy in addition the energy inequality (see Definition 3.4, τ_M can be estimated from below as follows: for all p there exists $N(p)$ with

$$
\mathbb{P}\left(\tau_M(\|X_0\|_V, \Phi^X) \ge t\right) \ge 1 - t^p N(p) \left(\frac{(1+\|X_0\|_H^4)(1+\|X_0\|_V^2)^2}{M}\right)^p
$$

For a proof see the paper [1].

Theorem 3.8. *(see* [1]*) Assume (S.2). Let X and $\tilde{X}$ be martingale solutions to* (15) *with some initial condition $X_0 = \tilde{X}_0 \in V$. Let $M > 0$ be a constant such that X and $\tilde{X}$ are strong solutions on the intervals* $[0, \tau_M(\|X_0\|_V, \Phi^X)[$ *and* $[0, \tau_M(\|X_0\|_V, \Phi^{\tilde{X}})[$ *respectively. Then:*

(i) $X \cdot 1_{[0,\tau_M(\|X_0\|_V, \Phi^X)[}$ *and* $\tilde{X} \cdot 1_{[0,\tau_M(\|X_0\|_V, \Phi^{\tilde{X}})[}$ *have the same laws.*

(ii) *If X and $\tilde{X}$ are defined on the same probability space, then* $[\tau_M(\|X_0\|_V, \Phi^X)[= [\tau_M(\|X_0\|_V, \Phi^{\tilde{X}})[$ *and $X = \tilde{X}$ on* $[0, \tau_M(\|X_0\|_V, \Phi^X)[$ $\mathbb{P}$*-a.s.*

The key tool for the analysis is the decomposition of (15) into the stochastic convolution

$$\Phi^u(t) = \int_0^t e^{(t-s)A} C(u_s) dW_s \,, \quad t \in [0,T]$$

and

$$Y_t := u_t - \Phi^u(t), \quad t \in [0,T].$$

(Y_t) then satisfies the evolution equation:

$$\dot{Y}_t = AY_t + B(Y_t + \Phi^u(t), Y_t + \Phi^u(t)) \,.$$

We therefore study the deterministic equation

$$\dot{y}_t = Ay_t + B(y_t + w_t) \,, \; w \in C([0,T]; H^{\frac{5}{4}}). \tag{17}$$

Proposition 3.9. *There exists M such that for all $u_0 \in V$ there exists a strong solution on the interval $[0, \tau_M(\|u_0\|_V, \varphi)[$.*

Proof: Let $(e_k)_{k\ge 1}$ be a CONS of H, consisting of eigenvectors of A, let $T_n(x) := \sum_{k=1}^n \langle x, e_k\rangle e_k$ be the orthogonal projection onto the space $H^n = \text{span}\{e_k \,:\, 1 \le k \le n\}$. Let u_n be the unique solution of the ordinary differential equation

$$\dot{y}_n(t) = Ay_n(t) + \Pi_n B(y_n(t) + w_n(t), y_n(t) + w_n(t)).$$

Then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y_n(t)\|_V^2 &= -\|Ay_n(t)\|_H^2 - \langle B(y_n(t)+w_n(t), y_n(t)+w_n(t)), Ay_n(t)\rangle_H \\
&\le -\|Ay_n(t)\|_H^2 + c_1\|y_n(t)+w_n(t)\|^2_{H^{\frac{5}{4}}}\|Ay_n(t)\|_H \\
&\le -\|Ay_n(t)\|_H^2 + c_1(\|y_n(t)\|^2_{H^{\frac{5}{4}}} + 2\|y_n(t)\|_{H^{\frac{5}{4}}}\|w_n(t)\|_{H^{\frac{5}{4}}} \\
&\qquad + \|w_n(t)\|^2_{H^{\frac{5}{4}}})\|Ay_n(t)\|_H \\
&\le -\|Ay_n(t)\|_H^2 + c_2\big(\|y_n(t)\|_V^{\frac{3}{2}}\|Ay_n(t)\|_H^{\frac{3}{2}} \\
&\qquad + 2\|y_n(t)\|_V^{\frac{3}{4}}\|w_n(t)\|_{H^{\frac{5}{4}}}\|Ay_n(t)\|_H^{\frac{5}{4}} \\
&\qquad + \|w_n(t)\|^2_{H^{\frac{5}{4}}}\|Ay_n(t)\|_H\big) \\
&\le -\frac{1}{2}\|Ay_n(t)\|_H^2 + c_3(\|y_n(t)\|_V^6 + \|w_n(t)\|^4_{H^{\frac{5}{4}}})
\end{aligned}$$

where we used the inequality

$$\|y\|_{H^{\frac{5}{4}}} \le c\|y\|_{H^1}^{\frac{3}{4}}\|y\|_{H^2}^{\frac{1}{4}}$$

in the second but last estimate. We conclude that in particular

$$\frac{1}{2}\frac{d}{dt}\left(1+\|y_n(t)\|_V^2\right) \le c_3\left(1+\|w_n(t)\|^4_{H^{\frac{5}{4}}}\right)\left(1+\|y_n(t)\|_V^2\right)^3$$

and thus

$$\|y_n(t)\|_V^2 \le \left(\frac{1}{1+\|y_n(0)\|_V^2} - c_4 \int_0^t (1+\|w_n(s)\|^4_{H^{\frac{5}{4}}}) ds\right)^{-\frac{1}{2}}.$$

We can now take the limit along some converging subsequence to conclude that any cluster point of $\{y_n\}$ converges weakly in $L^2(0,\tau,H^2)\cap C([0,\tau];V)$ for any $\tau < \tau_M(\|u(0)\|_V, w)$, $M = \frac{1}{c_4}$. □

Proof of Theorem 3.6: The existence of a (local) strong solution easily follows from Proposition 3.9 and Lemma 3.5. Indeed, consider the same Galerkin approximation as in the proof of 3.9 and let $X_n(t)$ be the solution of the SDE

$$dX_n(t) = [AX_n(t) + \Pi_n B(X_n(t), X_n(t))]dt + \Pi_n C(X_n(t))dW_t$$

Using the tightness method, we obtain a martingale solution $X(t)$ of (15) as a weak limit point of $X_n(t)$. Using the decomposition $X(t) = Y(t) + \Phi^X(t)$ we can first conclude from Lemma 3.5 that $\Phi^X \in C([0,\tau]; H^{\frac{5}{4}})$ and then from Proposition 3.9 that X is a strong solution on the interval $[0, \tau_M(\|u_0\|, \Phi^X)[$ □

Proof of Theorem 3.8

Step 1: Let $\varrho \in C_0^1(\mathbb{R})$, $1_{[0,1]} \le \varrho \le 1_{[0,2]}$ and define

$$\xi_n(Y)(t) := \varrho\left(n^{-1}\|Y(t)\|_V + n^{-1}\left(\int_0^t \|Y(s)\|^2_{H^2} ds\right)^{\frac{1}{2}}\right)$$

for

$$\begin{aligned} Y \in \mathcal{K}_{T,p} := \{&Y \in L^p(\Omega; L^2(0,T;H^2)) \cap L^p(\Omega; C([0,T];V)) \mid \\ &Y \text{ predictable w.r.t. } (\mathcal{F}_t)_{t\in[0,T]}\}. \end{aligned}$$

Then there exists $p > 2$ such that for all n and all $X_0 \in V$ there exists a unique solution $X^{(n)} \in \mathcal{K}_{T,p}$ of

$$\begin{aligned} X^{(n)}(t) = e^{tA}X_0 &+ \int_0^t e^{(t-s)A} B(X^{(n)}(s))\xi_n(X^{(n)})(s)\, ds \\ &+ \int_0^t e^{(t-s)A} C(X^{(n)}(s)) dW_s\,. \end{aligned}$$

Proof via fixed point argument:

$$\begin{aligned} \mathcal{K}^{(n)}(Y)(t) &= e^{tA}X_0 + \int_0^t e^{(t-s)A} B(Y(s))\xi_n(Y(s))\, ds \\ &\qquad + \int_0^t e^{(t-s)A} C(Y(s)) dW_s \\ &= e^{tA}X_0 + \mathcal{K}_1^{(n)}(Y)(t)(Y)(t) + \mathcal{K}_2^{(n)}(Y)(t)\,, \text{ say.} \end{aligned}$$

In the following, let

$$\|Y\|_{T,p} := \left(\mathbb{E}\left(\int_0^T \|Y(s)\|_{H^2}^2 \, ds \right)^{p/2} + \mathbb{E}\left(\sup_{t\in[0,T]} \|Y(t)\|_V^p \right) \right)^{\frac{1}{p}} .$$

Then there exist $T>0$ such that $\mathcal{K}^{(n)}(\mathcal{K}_{T,p}) \subset \mathcal{K}_{T,p}$ and $\mathcal{K}^{(n)}$ is a strict contraction. Indeed, for $Y,Z\in\mathcal{K}_{T,p}$ we can estimate

$$\begin{aligned}
&\|\mathcal{K}_1^{(n)}(Y)(t) - \mathcal{K}_1^{(n)}(Z)(t)\|_V \le \\
&\qquad \le C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|B(Y(s))\xi_n(Y(s)) - B(Z(s))\xi_n(Z(s))\|_H \, ds
\end{aligned}$$

and then for the integrand

$$\begin{aligned}
&\|B((Y)(s))\xi_n(Y)(s) - B(Z(s))\xi_n(Z)(s)\|_H \\
&\qquad \le \|B(Y(s))\xi_n(Y(s)) - B(Z(s))\xi_n(Z(s))\|_H(\xi_n(2^{-1}Y)(s) + \xi_n(2^{-1}Z)(s)) \\
&\qquad \le \{\|B(Y(s))\||\xi_n(Y(s)) - \xi_n(Z(s))|\xi_n(2^{-1}Y)(s) \\
&\qquad\qquad + \|B(Y(s)) - B(Z(s))\|\xi_n(Z(s))\xi_n(2^{-1}(Y(s))) \\
&\qquad\qquad + \|B(Z(s))\||\xi_n(Y(s)) - \xi_n(Z(s))|\xi(2^{-1}Z(s)) \\
&\qquad\qquad + \|B(Y(s)) - B(Z(s))\|\xi_n(Y(s))\xi_n(2^{-1}(Z(s)))\} \\
&\qquad \le C(n)\left\{ \|Y(s)\|_V^{\frac{3}{2}} \|Y(s)\|_2^{\frac{1}{2}} \xi_n(2^{-1}Y(s)) + \|Z(s)\|_V^{\frac{3}{2}} \|Z(s)\|_2^{\frac{1}{2}} \xi_n(2^{-1}Z(s)) \right\} \\
&\qquad\qquad \times \left\{ \|Y(s)-Z(s)\|_V + \left(\int_0^s \|Y(r)-Z(r)\|_2^2 dr \right)^{1/2} \right\}
\end{aligned}$$

where we used the estimates

$$\begin{aligned}
|B(u,u)| &\le C\|u\|_V^{\frac{3}{2}} \|u\|_2^{\frac{1}{2}} \\
|B(u,u) - B(v,v)| &\le C\|u-v\|_V\{\|u\|_V^{\frac{3}{2}}\|u\|_2^{\frac{1}{2}} + \|v\|_V^{\frac{3}{2}}\|v\|_2^{\frac{1}{2}}\} .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]} \|\mathcal{K}_1^{(n)}(Y)(t) - \mathcal{K}_1^{(n)}(Z)(t)\|_V^p \right) \\
&\quad \le C(n)\|Y-Z\|_{T,p}^p \Big(\sup_{t\in[0,T]} \int_0^t (t-s)^{-\frac{1}{2}} \|Y(s)\|_2^{\frac{1}{2}} \xi_n(2^{-1}Y)(s)\, ds \\
&\qquad + \int_0^t (t-s)^{-\frac{1}{2}} \|Z(s)\|_2^{\frac{1}{2}} \xi_n(2^{-1}Y)(s)\, ds \Big)^p \\
&\quad \le C(n)\|Y-Z\|_{T,p}^p \left(\int_0^t (t-s)^{-\frac{4}{6}} ds \right)^{\frac{3p}{4}} \cdot \left(\int_0^t \|Y(s)\|_2^2 1_{\{\int_0^s \|Y(r)\|_2^2 dr \le 4n^2\}} \right)^{\frac{p}{4}} \\
&\quad \le C(n) t^{\frac{p}{4}} \|Y-Z\|_{T,p}^p .
\end{aligned}$$

Similarly,

$$\mathbb{E}\left(\int_0^T \|\mathcal{K}_1^{(n)}(Y)(t)-\mathcal{K}_1^{(n)}(Z)(t)\|_2^2\,dt\right)^{\frac{p}{2}} \le C(n)T^{\frac{1}{6}}\|Y-Z\|_{T,p}.$$

The Lipschitz contraction for $\mathcal{K}_2^{(n)}$ can be shown similarly, using (S.2). This proves Step 1.

Step 2: Let X be a martingale solution to (15) defined on the same probability space as each $X^{(n)}$. Let M be such that X is strong on $[0,\tau_M(\|X(0)\|_V,\Phi^X)[$ and let

$$\tau_n := \inf\{t\ge 0 : \|X(t)\|_V + \left(\int_0^t \|X(s)\|_2^2 ds\right)^{\frac{1}{2}} \le n\}\wedge T\,.$$

Then

$$X = X^{(n)} \text{ on } [0,\tau_n\wedge\tau_M(\|X(0)\|_V,\Phi^X)[\,.$$

Let $\tilde{X}$ be a different martingale solution, possibly defined on a different probability space, then also

$$\tilde{X} = \tilde{X}^{(n)} \text{ on } [0,\tilde{\tau}_n\wedge\tau_M(\|X(0)\|_V,\Phi^{\tilde{X}})[$$

Now, the laws of $X^{(n)}$ and $\tilde{X}^{(n)}$ coincide, hence also the laws of $\tau_n,\tilde{\tau}_n$. Therefore the laws of

$$1_{[0,\tau_n\wedge\tau_M(\dots)[}X \text{ and } 1_{[0,\tilde{\tau}_n\wedge\tau_M(\dots)[}\tilde{X}$$

coincide for all n. Letting $n\to\infty$ we obtain (i).

The proof of (ii) is obvious. □

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