Model Theory

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Exercises No.6

Exercise 1 [ω-saturation]
Show that $\mathrm{Th}(\mathfrak{N})$ does not have countable ω-saturated models.

Exercise 2 [no atomic models]
Let $\tau = \{P_i \colon i \in \mathbb{N}\}$ with unary predicates P_i and consider the theory

$$T := \Big\{\exists x\big(\bigwedge_{i\in s} P_i x \wedge \bigwedge_{i\in t} \neg P_i x\big) \colon s, t \subseteq \mathbb{N}, s \cap t = \emptyset\Big\}.$$

Show that T is complete, that $S_1(T)$ is uncountable, and that T has no isolated types. In which cardinalities does T have κ-saturated models?

Exercise 3 [equivalence relations]
Investigate the completions and the type spaces of the (incomplete) theory of one equivalence relation. A good starting point is the classification of all countable models up to isomorphism and up to elementary equivalence.
What are atomic and ω-saturated models?

Exercise 4 [just k countable models]
Consider $\tau = \{<\} \cup \{c_n \colon n \in \mathbb{N}\}$ and the τ-structures $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$ obtained as expansions of $(\mathbb{Q}, <)$ by interpretation of the constants according to

$$c_n^{\mathfrak{A}_1} = n \qquad c_n^{\mathfrak{A}_2} = -1/n \qquad c_n^{\mathfrak{A}_3} = (1 - 1/n)^n$$

(i) Show that $\mathfrak{A}_1 \equiv \mathfrak{A}_2 \equiv \mathfrak{A}_3$.
(ii) Show that every countable $\mathfrak{B} \equiv \mathfrak{A}_i$ is isomorphic to precisely one of these $\mathfrak{A}_i$.
(iii) Investigate which 1-types of T are realised in these models. Is any one of them ω-saturated, is any one of them atomic?
(iv) Can you think of a recipe based on the above to cook up a complete first-order theory with precisely k non-isomorphic countable models for every $k \geqslant 3$.