Model Theory

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Exercises No.5

Exercise 1 [for set theorists: end extensions]
An elementary *end extension* of a model of set theory ZF, $\mathfrak{A} = (A, \in^{\mathfrak{A}})$, is a proper elementary extension $\mathfrak{B} \succcurlyeq \mathfrak{A}$ in which no member of $A \subseteq B$ gets new elements in $\mathfrak{B}$: f.a. $a \in A$, $\{b \in B \colon b \in^{\mathfrak{B}} a\} \subseteq A$.
Show that every countable $\mathfrak{A} \models$ ZF admits an elementary end extension. The crucial step is this. For a new constant c put $T := \mathrm{D}(\mathfrak{A}) \cup \{\neg\, c \in c_a \colon a \in A\}$. We are looking for a model of T that omits all types of the form $p_a := \{x \in c_a\} \cup \{\neg\, x = c_d \colon d \in a\}$.

It remains to show that none of these types is isolated in $S_1(T)$.

Show that $\varphi(x, c) \in \mathrm{FO}[\{\in, c\}_A]$ is consistent with T iff

$$\mathrm{D}(\mathfrak{A}) \models \forall z \exists y \exists x (\neg\, y \in z \wedge \varphi(x, y)).$$

From this one can use standard set theoretic (ZF) reasoning in $\mathfrak{A}$ to show that, if $T, \varphi(x, c) \models x \in c_a$, then for some $d \in A$, $\varphi(c_d, c) \wedge c_d \in c_a$ is consistent with T, whence $\varphi(x, c)$ does not isolate p_a. Then Corollary 1.13 yields a model as desired.

Exercise 2 [saturation properties of ultrapowers]
Let $\mathfrak{A}$ be a τ-structure, $\mathcal{U}$ a non-principal ultrafilter over $\mathbb{N}$ and $\mathfrak{A}^* := \mathfrak{A}^{\mathbb{N}}/\mathcal{U}$ the corresponding ultrapower. Recall that we may think of $\mathfrak{A}^*$ as an elmentary extension $\mathfrak{A} \preccurlyeq \mathfrak{A}^*$, by identifying $\mathfrak{A}$ with its natural embedding into $\mathfrak{A}^*$.

(a) Show that $\mathfrak{A}^*$ realises every countable partial type with parameters from A^*.

(b) Conclude that for countable signatures τ and every countable τ-structure $\mathfrak{A}$, the ultrapower $\mathfrak{A}^*$ is ω_1-saturated.

NB: (a) requires not just the realisation of partial types with parameters from A (corresponding to elements of A^* that are represented by constant sequences in $A^{\mathbb{N}}$). You may want to consider this simpler case of parameters just from A first – that proof is very much the same as for (countable) compactness.

Exercise 3 [κ-saturation]
Show the criterion for κ-saturation ($\kappa \geqslant \omega$) is not actually weakened if only expressed for 1-types (with parameters) instead of n-types. Provide, e.g., the argument that shows that all 2-types with fewer than κ parameters are realised if we know that all 1-types with fewer than κ parameters are realised.

Exercise 4 [dense linear orderings]
We consider dense linear orderings.

(a) Show that the theory $T := \mathrm{Th}(\mathbb{Q}, <)$ of dense linear orderings without endpoints has *quantifier elimination*, i.e., for every $\varphi(\mathbf{x}) \in \mathrm{FO}_n[<]$ there is some quantifier-free $\varphi'(\mathbf{x})$ such that $T \models \forall \mathbf{x}(\varphi \leftrightarrow \varphi')$. What does this say about $S_n(T)$?

(b) Show that $(\mathbb{Q}, <)$ is ω-saturated (a countable, saturated model of T).

(c) Let κ be some infinite cardinal. Show that a dense linear ordering $\mathfrak{A} = (A, <)$ without endpoints is κ-saturated if, and only if, it realises every *cut* of cardinality less than κ. Here a cut (of cardinality less than κ) in $(A, <)$ is a pair of subsets $L, U \subseteq A$ (of cardinality less than κ) such that $\ell < u$ for all $\ell \in L, u \in U$; this cut is *realised* by an element $a \in A$ if $a < u$ for all $u \in U$ and $\ell < a$ for all $\ell \in L$.

(d) Use (c) to generalise the countable categoricity of T (cf. exercise 4 on sheet 4) as follows: any two κ-saturated models of the same cardinality κ of T are isomorphic.

(e) Check your argument for (d) as to its scope: can you modify it to show that any two saturated, elementarily equivalent τ-structures of the same cardinality are isomorphic?