Model Theory

Martin Otto
Summer 2010

Exercises No.4

Exercise 1 [back&forth systems; Lemma 3.3]
Show by induction on $m \in \mathbb{N}$ that any partial isomorphism π in the m-th level of some (graded) back&forth system between relational structures $\mathfrak{A}$ and $\mathfrak{B}$ preserves all FO-formulae up to quantifier rank m.

Exercise 2 [review of Ehrenfeucht–Fraïssé, Theorem 3.4]
Review the Ehrenfeucht–Fraïssé Theorem and its proof for levels $\simeq_m$ and $\simeq_{\mathrm{fin}}$.
Discuss the relationship with $\simeq_{\mathrm{part}}$ and give examples of pairs of structures that separate these levels of back&forth equivalence.
Why is finiteness of τ important? Give (counter)examples.

Exercise 3 [non-relational signatures]
With an arbitrary signature τ associate the relational signature τ_{Rel} consisting of the relation symbols of τ together with a new unary relation symbol R_c for each $c \in \tau$ and a new $(n+1)$-ary relation symbol R_f for each n-ary $f \in \tau$.

For a τ-structure $\mathfrak{A}$ let $\mathfrak{A}_{\mathrm{Rel}}$ be the τ_{Rel}-structure in which functions $f^{\mathfrak{A}}$ are encoded by their graphs (as $R_f^{\mathfrak{A}_{\mathrm{Rel}}}$) and constants $c^{\mathfrak{A}}$ by their singleton sets (as $R_c^{\mathfrak{A}_{\mathrm{Rel}}}$).

(a) Axiomatise the class of all $\mathfrak{A}_{\mathrm{Rel}}$ as a class of τ_{Rel}-structures.

(b) Provide a translation function for FO[τ]-formulae, $\varphi \longmapsto \varphi_{\mathrm{Rel}}$ s.t. for all $\mathfrak{A}, \mathbf{a}$:

$$\mathfrak{A}, \mathbf{a} \models \varphi \quad \Leftrightarrow \quad \mathfrak{A}_{\mathrm{Rel}}, \mathbf{a} \models \varphi_{\mathrm{Rel}}.$$

(c) How does the quantifier rank of φ_{Rel} depend on φ? Formulate a variant of the Ehrenfeucht–Fraïssé Theorem for structures in arbitrary (rather than relational) finite signatures τ w.r.t. a suitable gradation of $\equiv$.

Exercise 4 [a back&forth construction]
Show that any two countable dense linear orderings without endpoints are isomorphic. Conclude that the FO theory of dense linear orderings is complete (and countably categorical). Show that there are dense linear orderings of the same uncountable cardinality that are not isomorphic (though elementarily equivalent).
What is the situation for dense linear orderings without endpoints with a unary predicate for a dense subset?