Model Theory | MARTIN OTTO
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Exercises No.3

Exercise 1 [variation on Tarski's thm and Lemma 2.12]
Show (the expressive completeness claim) of Tarski's theorem in the following variants. Use the fragment $\Delta := \forall\text{-FO}_0[\tau] \subseteq \text{FO}_0[\tau]$ and the associated notion of $\forall$-transfer $\mathfrak{A} \Rightarrow_\forall \mathfrak{B}$.

(a) T.f.a.e. for any (satisfiable) $\varphi \in \text{FO}_0[\tau]$:
  (i) φ preserved under $\Rightarrow_\forall$
  (ii) $\varphi \equiv \varphi' \in \forall\text{-FO}[\tau]$.

(b) T.f.a.e. for any (satisfiable) $T \subseteq \text{FO}_0[\tau]$:
  (i) T preserved under $\Rightarrow_\forall$
  (ii) T is axiomatised by some $T_\forall \subseteq \forall\text{-FO}_0[\tau]$ (so that $T_\forall \models T$ and $T \models T_\forall$).

Which closure conditions of Δ are relevant for each of these proof patterns? (Of course, $\Delta = \forall\text{-FO}_0[\tau]$ is closed under positive boolean combinations, but what matters where?)

The following two exercises lead up to a classical theorem linking preservation in limits of chains to prenex $\forall$-$\exists$-axiomatisability.

Exercise 2 [an extension configuration]
Consider the fragment $\Delta \subseteq \text{FO}_0[\tau]$ consisting of all $\forall^*\exists^*$-sentences in $\text{FO}[\tau]$, i.e., all prenex sentences of the form $\forall\mathbf{x}\exists\mathbf{y}\chi(\mathbf{x},\mathbf{y})$ with a quantifier-free kernel $\chi \in \text{FO}[\tau]$.

We define $\forall^*\exists^*$-transfer $\mathfrak{A} \Rightarrow_{\forall^*\exists^*} \mathfrak{B}$ accordingly.

Show that transfer $\mathfrak{A} \Rightarrow_{\forall^*\exists^*} \mathfrak{B}$ between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ implies the existence of τ-structures $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{B}' \succcurlyeq \mathfrak{B}$ such that $\mathfrak{B} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}'$.

Hint: work with FO-theories, algebraic and (the universal parts of) elementary diagrams as appropriate to specify suitable $\mathfrak{A}'_B$ and $\mathfrak{B}'_{A'}$.

Exercise 3 [$\forall^*\exists^*$ preservation and limits of chains]
A sentence $\varphi \in \text{FO}_0[\tau]$ is said to be preserved under limits of chains if, whenever $(\mathfrak{A}_\alpha)_{\alpha<\gamma}$ is a chain of τ-structures with limit $\mathfrak{A} := \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$, then

$$\mathfrak{A}_\alpha \models \varphi \text{ for all } \alpha < \gamma \quad \text{implies} \quad \mathfrak{A} \models \varphi.$$

(a) Show that $\forall^*\exists^*$-sentences are preserved under limits of chains.

(b) Show that preservation of $\varphi \in \text{FO}_0[\tau]$ under limits of chains implies its preservation under $\Rightarrow_{\forall^*\exists^*}$.

(c) Conclude that t.f.a.e. for any (satisfiable) $\varphi \in \text{FO}_0[\tau]$:
  (i) φ is preserved under limits of chains;
  (ii) $\varphi \equiv \varphi'$ where φ' is an $\forall^*\exists^*$-sentence.

see back for hints for Exercise 3

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(a) Show that $\forall^*\exists^*$-sentences are preserved under limits of chains.
(b) Show that preservation of $\varphi \in \mathrm{FO}_0[\tau]$ under limits of chains implies its preservation under $\Rightarrow_{\forall^*\exists^*}$.
(c) Conclude that t.f.a.e. for any (satisfiable) $\varphi \in \mathrm{FO}_0[\tau]$:
  (i) φ is preserved under limits of chains;
  (ii) $\varphi \equiv \varphi'$ where φ' is an $\forall^*\exists^*$-sentence.

Hint for (b): for $\mathfrak{A} \Rightarrow_{\forall^*\exists^*} \mathfrak{B}$ use the extension configuration from Exercise 2 repeatedly to construct an elementary ω-chain

$$\mathfrak{B} = \mathfrak{B}_0 \preccurlyeq \mathfrak{B}_1 \preccurlyeq \cdots$$

with an interleaving ω-chain of structures $\mathfrak{A}_i \equiv \mathfrak{A}$,

$$\mathfrak{B}_0 \subseteq \mathfrak{A}_0 \subseteq \mathfrak{B}_1 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \subseteq \cdots$$

Note that these chains have the same limits $\mathfrak{A}_\omega = \mathfrak{B}_\omega$.
If $\mathfrak{A} \models \varphi$ and if φ is preserved under chains , then $\mathfrak{A}_\omega \models \varphi$ implies that also

$$\mathfrak{B} = \mathfrak{B}_0 \preccurlyeq \mathfrak{B}_\omega = \mathfrak{A}_\omega \models \varphi.$$

Hint for (c): adapt the proof idea from Lemma 2.12 and Exercise 1 above to the case of the $\forall^*\exists^*$-fragment as Δ, which, although not closed under conjunction and disjunction literally, is closed in this sense up to logical equivalence.